Loci of Kimberling Centers in the Thales Configuration: An Analytic and Algebraic Approach

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Abstract

The Encyclopedia of Triangle Centers catalogs thousands of triangle centers defined in trilinear or barycentric coordinates, yet the geometric behavior of these centers under continuous deformation of the reference triangle remains comparatively unexplored. This paper studies the loci of selected Kimberling centers within a constrained geometric setting known as the Thales configuration, where one side of the triangle is fixed as a diameter of a circle and the third vertex varies along the corresponding semicircle. Using barycentric and trilinear coordinate representations, we derive explicit Euler-line parametrizations for families of triangle centers and compute the resulting loci in Cartesian coordinates via symbolic elimination. The algebraic curves obtained are analyzed using tools from algebraic geometry, including degree, singularity structure, and genus. In many cases the loci admit rational parametrizations, revealing that the geometric motion of these centers corresponds to genus-zero algebraic curves. This framework provides a unified analytic and algebraic description of triangle center loci and illustrates how classical triangle geometry naturally interfaces with modern methods from computational algebraic geometry.

1 Introduction

The study of triangle centers has undergone a renaissance in the digital age, largely due to the exhaustive categorization provided by Clark Kimberling's *Encyclopedia of Triangle Centers* (ETC). While the ETC provides a rigorous framework for the trilinear and barycentric definitions of over 60,000 points, these definitions are typically presented in a static context relative to a fixed reference triangle.

This paper explores the **dynamic evolution** of these centers. By constraining a family of right triangles within a fixed circumcircle, we treat the triangle's vertices as functions of a single angular parameter θ. As the vertex C traverses the arc of the circle, the associated Kimberling centers X_n trace continuous paths—or loci—in the Euclidean plane.

The objective of this work is twofold:

1. To derive the Cartesian equations for the loci of selected centers, including the Nine-Point Center (X_5) and the Feuerbach Point (X_{11}).

2. To bridge the gap between abstract coordinate geometry and visual intuition through interactive dynamic geometry software.

By focusing on the inscribed right triangle, we isolate specific algebraic behaviors that are often obscured in more general configurations, revealing a hidden symmetry within the ETC's classification.

2 Foundations

To ensure the study is invariant under scaling, we define the circumcircle as a unit circle centered at the origin $O(0,0)$ in the Cartesian plane:

$$x^2 + y^2 = 1$$

Consider the right triangle $\triangle ABC$ where the hypotenuse $\left|\overline{AB}\right|$ is a diameter fixed along the x-axis. The vertices are defined as:

- $A = (1, 0)$
- $B = (-1, 0)$
- $C = (\cos\theta, \sin\theta)$, where $\theta \in [0, 2\pi)$ represents the vertex position on the circumference of the circle

It can be shown that in the Thales configuration,

$$\begin{aligned}
c &= \left|\overline{AB}\right| = \text{Diameter of a unit circle} = 2 \\
b &= \left|\overline{AC}\right| = \text{The length of the side opposite to angle } \angle B = 2\left|\cos\left(\frac{\theta}{2}\right)\right| \\
a &= \left|\overline{BC}\right| = \text{The length of the side opposite to angle } \angle A = 2\sin\left(\frac{\theta}{2}\right)
\end{aligned}$$

As side lengths are strictly non-negative in traditional geometry, absolute value brackets are included. However, the absolute values break analyticity at $\theta = \pi k$, where k is an integer. Therefore, to preserve analytical parametrizations, we remove the absolute value by focusing on the upper plane, where if $\theta \in [0, \pi)$, then

$$a = 2\sin\left(\frac{\theta}{2}\right), \qquad b = 2\cos\left(\frac{\theta}{2}\right), \qquad c = 2$$

Another fundamental set of units for finding center loci are interior angles of $\triangle ABC$, where by the half-angle geometry,

$$\begin{aligned}
\gamma =& \angle ACB = \frac{\pi}{2} \\
\alpha =& \angle CAB = \frac{1}{2}(\pi - \theta) \\
\beta =& \angle CBA = \frac{\theta}{2}
\end{aligned}$$

2.1 Trilinear and Barycentric Coordinates

To locate a center X_n, we utilize two primary homogeneous coordinate systems defined relative to $\triangle BCD$:

1. **Trilinear coordinates:** Coordinates in the form of $(x : y : z)$ represent ratios of the directed distances from the point to the sides a, b, and c. For a right triangle, these are often functions of $\sin\left(\frac{\theta}{2}\right)$ and $\cos\left(\frac{\theta}{2}\right)$. Trilinear coordinates satisfy the following

$$ah_a + bh_b + ch_c = 2 \cdot \text{Area}(\triangle ABC)$$

   where (h_a, h_b, h_c) denote the *absolute trilinear coordinates.*

2. **Barycentric coordinates:** Coordinates in the form of $(\lambda_1 : \lambda_2 : \lambda_3)$ represent the ratio of the areas of the sub-triangles formed by the point and the vertices. They are related to trilinears by the side lengths:

$$(\lambda_1 : \lambda_2 : \lambda_3) = (xa : yb : zc)$$

While trilinears can compute aesthetically and efficiently, barycentrics are required for rectangular-coordinate mapping. The locus of the center X_n is computed by the weighted average of the vertices:

$$X_n(\theta) = \frac{\lambda_1 A + \lambda_2 B + \lambda_3 C}{\lambda_1 + \lambda_2 + \lambda_3}$$

Substituting our vertex coordinates yields

$$X_n(\theta) = \left(\frac{\lambda_1 - \lambda_2 + \lambda_3 \cos\theta}{\lambda_1 + \lambda_2 + \lambda_3}, \frac{\lambda_3 \sin\theta}{\lambda_1 + \lambda_2 + \lambda_3} \right) \tag{1}$$

Above can also be derived, using trilinear coordinates. Given a reference triangle $\triangle ABC$, for side lengths a, b and c:

$$(x : y : z) = \left(\frac{k_1}{a} : \frac{k_2}{b} : \frac{1 - k_1 - k_2}{c} \right)$$

where $\vec{X_n} = k_1\vec{A} + k_2\vec{B}$ is a vector of a point P in the Cartesian plane, and

$$k_1 = \frac{ax}{ax + by + cz}, \qquad k_2 = \frac{by}{ax + by + cz}$$

With $(0, 0)$ chosen and vertex coordinates A, B and C known, which can be presented as vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$, then

$$\vec{X_n} = \frac{ax}{ax + by + cz}\vec{A} + \frac{by}{ax + by + cz}\vec{B} + \frac{cz}{ax + by + cz}\vec{C}$$

where

$$|\vec{C} - \vec{B}| = a,$$
$$|\vec{A} - \vec{C}| = b,$$
$$|\vec{B} - \vec{A}| = c.$$

Next sections demonstrate examples of their applications.

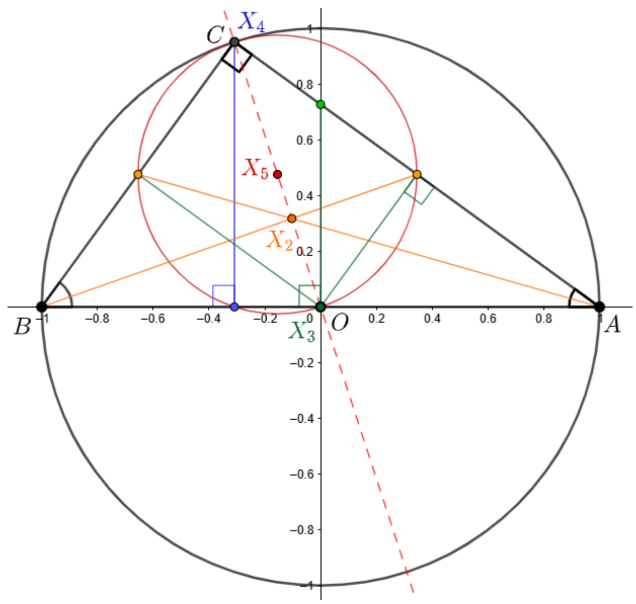

Figure 1: Euler's line in red, medians in orange, altitudes in blue, perpendicular lines in green

2.2 Structural Invariant: The Euler Line

Another important component of finding the center loci is the Euler line, which acts as a central line governed by essential points, including X_5 (nine-point center). Several important points that Euler discovered are X_2, X_3 and X_4, which are respectively centroid, circumcenter and orthocenter. As shown, these points lie on the Euler line denoted by the red dashed line. In the case of right triangles:

- In a right triangle the circumcenter coincides with the midpoint of the hypotenuse and the orthocenter coincides with the right-angle vertex. Consequently, the Euler line is the radius $\overline{OC}$;
- Point X_3 coincides with point O. Their trilinear coordinates are $(\cos\alpha : \cos\beta : \cos\gamma) = (\cos\alpha : \cos\beta : 0)$, and their barycentric coordinates are $(\sin 2\alpha : \sin 2\beta : \sin 2\gamma) = \left(\frac{1}{2} : \frac{1}{2} : 0\right)$, which each imply that X_3 is $(0, 0)$, the midpoint of $\overline{AB}$;
- Point X_4 coincides with point C. Their trilinear coordinates are $(\sec\alpha : \sec\beta : \sec\gamma) \equiv (0 : 0 : 1)$, where $\sec\gamma$ does not exist for $\gamma = \frac{\pi}{2}$. Alternatively, their barycentric coordinates are $(\tan\alpha : \tan\beta : \tan\gamma) \equiv (0 : 0 : 1)$, which shows that $X_4 = (\cos\theta, \sin\theta)$;

2.3 Euler Line Representations

From these observations, we can establish that if $(x : y : z)$ is a variable point in trilinear coordinates, the equation of the Euler line is generally

$$\cos\theta y = \sin\theta x$$

following that

$$\sin(2\alpha)\sin(\beta-\gamma)x + \sin(2\beta)\sin(\gamma-\alpha)y + \sin(2\gamma)\sin(\alpha-\beta)z = 0$$

Alternatively, in barycentric coordinates, setting $\lambda_1 = \lambda_2$ and $\lambda_3 = 0$ in Equation (1) yields

$$X_n(\theta) = \left(\frac{\lambda_3\cos\theta}{2\lambda_1+\lambda_3}, \frac{\lambda_3\sin\theta}{2\lambda_1+\lambda_3}\right) \equiv (\cos\theta, \sin\theta)$$

which yields the same result by taking the ratio of y-coordinates to x-coordinates. In the Thales configuration, the equation collapses to a radial vector field.

2.4 Algebraic Geometry of Loci

In algebraic geometry, an **affine algebraic curve**, or simply algebraic curve, is the set of points (x, y) in a Cartesian plane that satisfies the polynomial constraint $F(x, y) = 0$, where $F(x, y)$ denotes a polynomial function. In the context of the Thales configuration, the path traced by a triangle center X_n is not merely a set of points, but a real **algebraic variety** denoted by $\mathcal{V} \subset \mathbb{R}^2$. The aim of this subsection is to show that the traced paths are algebraic curves. It complements the parametric trace and its implicit stability by algebra geometry, which helps analyze the nuance of the center loci.

2.4.1 Polynomial Elimination

One of the methods to evaluate the locus is **implicitization**, which eliminates the parameter θ from the system of equations. A triangle center locus $X_n(\theta) = (x(\theta), y(\theta))$, which is a mapping, is given by the following system of equations

$$\begin{cases} x = f(\cos\theta, \sin\theta) \\ y = g(\cos\theta, \sin\theta) \\ \cos^2\theta + \sin^2\theta = 1 \end{cases}$$

Formally, it is defined as the zero-set of an implicit polynomial function $F(x, y) \in \mathbb{R}[x, y]$:

$$\mathcal{L}(X_n) = \left\{(x, y) \in \mathbb{R}^2 \middle| F(x, y) = 0\right\}$$

To treat this as a variety, let $u = \cos\theta$ and $v = \sin\theta$. The locus is the projection onto the (x, y)-plane of the intersection of these surfaces in $\mathbb{R}^4$ (or $\mathbb{C}^4$ since $\mathbb{R}^4 \subset \mathbb{C}^4$). Mathematically, we define that as $\pi : S \to \mathbb{R}^4$, where

$$S = \left\{(x, y, u, v) : x = f(u, v), y = g(u, v), u^2 + v^2 = 1\right\}$$

is the set defined in the affine space $\mathbb{A}^4$. According to Chevalley's Theorem, this process guarantees that the projection $\pi(S)$ is a constructible subset of $\mathbb{R}^2$.

Theorem 1 (Chevalley's Theorem for Constructible Sets)**.** *Let $f : X \to Y$ denote a finitely presented morphism of schemes. If $Z \subset X$ is a locally constructible subset, then $f(Z)$ is also locally constructible in Y.*

Another method to find the implicit function $F(x, y) = 0$ uses **resultants**, or alternatively the eliminants. The resultant of two polynomials is defined as a polynomial determined by the following system of equations

$$\begin{cases} h_1(t) = x \cdot R(t) - P(t) = 0 \\ h_2(t) = y \cdot R(t) - Q(t) = 0 \end{cases}$$

where the variety is given by

$$F(x, y) = \operatorname{Res}(h_1, h_2, t) = 0$$

Corollary 1. *The resultant vanishes if and only if the two polynomials share a common zero in t, which encodes the existence of parameter t.*

The more modern framework utilizes **Gröbner bases** for complex triangle centers, like X_{11}, that involve heavy computations. Under a lexicographical ordering, we compute the ideal defined by

$$I = \langle x \cdot R(t) - P(t), y \cdot R(t) - Q(t) \rangle$$

By computing the Gröbner basis and eliminating t, we find the implicit equation that lies in $I \cap \mathbb{R}[x, y]$. Once $F(x, y) = 0$ is obtained, the locus can then be classified by following tools of algebraic geometry:

- **Irreducibility:** If $F(x, y)$ cannot be factored into lower-degree polynomials, the variety is irreducible.
- **Singularities:** Points, where $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ both vanish, indicate nodes, cusps or self-introductions. Mathematically, the singularity criterion specifies

  $$\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right) = (0, 0)$$

  which correctly identifies these loci points.
- **Ideal Membership:** We can check if a center X_n always lie on a known curve by checking if the polynomial, defining the curve, belongs to the ideal of the center's locus.

2.4.2 Algebraic Varieties of Triangle Center Loci

For triangle centers that lie on the Euler line, we can establish the following:

Theorem 2. *Let $X_n(\theta)$ denote a triangle center, lying on the Euler line of a nondegenerate right triangle $\triangle ABC$ inscribed in a unit circle. Then, there exists a scalar function $\lambda_n(\theta)$, such that*

$$X_n(\theta) = \lambda_n(\theta)\,(\cos\theta, \sin\theta)$$

In particular, every Euler-line center in this configuration admits a polar representation in the form $r(\theta) = \lambda_n(\theta)$, which is equivalent to the parametric curve in 1-dimensional subspace

$$\{t(\cos\theta, \sin\theta) : t \in \mathbb{R}\}$$

The Euler-line points can be generally characterized as follows:

Theorem 3. *Let X_n lie on the Euler line. Then, there exists a scalar $\lambda_n(\theta)$, such that*

$$X_n(\theta) = \lambda_n(\theta)C(\theta)$$

where $C(\theta)$ denotes the parametric curve that spans X_n.

Moreover, two non-overlapping triangle centers generates a line. Generally, in a degenerate triangle $\triangle DEF$, the Euler line is defined by the affine line

$$\mathcal{E} = \{P_1 + t(P_2 - P_1) : t \in \mathbb{R}\}$$

where P_1 and P_2 are distinct non-overlapping points.

Theorem 4 (Scalar Uniqueness)**.** *Let X_n denote a triangle center lying on the Euler line. If P_1 and P_2 are two non-overlapping points, there exists a unique scalar $\lambda_n \in \mathbb{R}$, such that*

$$X_n = P_1 + \lambda_n(P_2 - P_1)$$

Proof. Suppose that λ_1 and λ_2 are scalars, such that $\lambda_1 \neq \lambda_2$. Since $P_1 \neq P_2$, then we have

$$(\lambda_1 - \lambda_2)(P_1 - P_2) = 0$$

In the $\mathbb{R}^2$ plane, since $P_2 - P_1 \neq 0$, then $\lambda_1 = \lambda_2$, contradicting the assumption. □

Corollary 2 (Homothety; Shape-Invariant)**.** *The locus of X_n is the circle:*

$$x^2 + y^2 = k^2$$

if and only if $\lambda_n(\theta) = k \in \mathbb{R}$ is a constant. Alternatively, the polar curve is $r(\theta) = |\lambda_n|$.

Corollary 3 (Shape-Variant)**.** *If $\lambda_n(\theta)$ is nonconstant, then the locus is the polar curve $r(\theta) = \lambda_n(\theta)$.*

Corollary 4. *If $\lambda_n(\theta)$ is rational in $\sin\left(\frac{\theta}{2}\right)$ and $\cos\left(\frac{\theta}{2}\right)$, then the locus is an algebraic curve.*

Proof. These hold since in the Thales configuration, side lengths are given by $2\cos\left(\frac{\theta}{2}\right)$ and $2\sin\left(\frac{\theta}{2}\right)$. By the Weierstraß substitution $\tan\left(\frac{\theta}{2}\right) \mapsto t$, where

$$(\cos\theta, \sin\theta) \mapsto \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$$

then $X_n(\theta)$ can be expressed as rational functions

$$X_n(\theta) = \left(\frac{P(t)}{R(t)}, \frac{Q(t)}{R(t)}\right)$$

which is defined as $X_n : \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$, a rational map from a 1-dimensional variety to a 2-dimensional variety. Thus, since the parameter space is a rational curve, the algebraic elimination ensures that the **Zariski's closure** restores X_n as the full algebraic curve, which we denote that as $\overline{X_n(\mathbb{P}^1)}$. □

In a formal research context, these corollaries can be unified into the following theorem:

Theorem 5 (Euler-Line Polar Parametrization Theorem)**.** *Let X_n denote a triangle center that lies on the Euler line of a right triangle $\triangle ABC$ in a unit circle. The locus of X_n is an algebraic variety of genus zero, whose polar representation is $r = |\lambda_n(\theta)|$. The variety is a circle if and only if the center is shape-invariant.*

Remark. *Since barycentric or trilinear coordinates can vanish in degeneracies, the rational map may not be defined everywhere. Moreover, some computer-generated plotting softwares do not "see" these gaps as $t \to +\infty$.*

2.5 Topological Invariants of the Resulting Algebraic Curves

The algebraic and topological classification of plane curves relies on classical results concerning rational parametrizations, singularities, and genus. We briefly recall the necessary notions here; standard references include [1, 3, 5].

As covered, in the given Thales representation, every triangle center X_n, lying on the Euler line, admits a representation

$$X_n(\theta) = \lambda_n(\theta)\, H(\theta)$$

where $H(\theta) = C\,(\cos\theta, \sin\theta)$ is the orthocenter of the triangle, and $\lambda_n(\theta)$ is the associated *Euler scalar function*. In polar coordinates relative to the circle's center, the locus therefore takes the form

$$r = |\lambda_n(\theta)|.$$

The algebraic and topological properties of these loci, which are related, are governed primarily by the analytic structure of $\lambda_n(\theta)$. Once the loci of triangle centers are expressed as algebraic curves, their global behavior can be analyzed using topological invariants. Among these, the genus plays a central role, measuring the intrinsic complexity of the curve independently of its embedding in the plane.

2.5.1 Arithmetic and Geometric Genus

A fundamental result in algebraic geometry states that any plane algebraic curve admitting a non-constant rational parametrization is a rational curve. Equivalently, its function field is isomorphic to the rational function field $k(t)$, and therefore the geometric genus of the curve is 0 (see [1, Ch. 1], [5]). In the simplified perspective, the genus is a unit that counts the number of "holes" of a connected surface or curve. Let's recall several algebraic concepts that govern the topology of curves:

Definition 1 (Algebraic Curve)**.** *An algebraic curve C over a field K is a one–dimensional algebraic variety defined as the zero set of a polynomial*

$$F(x, y) \in K[x, y].$$

Its associated function field $K(C)$ consists of all rational functions defined on the curve.

The algebraic curve can be defined alternatively as follows:

Definition 2. *An algebraic curve C defined over a field k is called rational if its function field is isomorphic to the rational function field $k(t)$.*

Definition 3 (Algebraic Geometry)**.** *The genus g of an algebraic curve is a topological invariant equal to the number of handles of the associated compact Riemann surface.*

Theorem 6 (Degree–Genus Formula, classical)**.** *The geometric genus of an irreducible plane curve of degree d is*

$$g = \frac{(d-1)(d-2)}{2} - \sum \delta_P$$

where $\sum \delta_P$ denote the delta invariants of singularities.

A proof may be found in [1, §3] or [3, Ch. IV].

Remark. *Arithmetic genus measures the degree-based complexity of the polynomial, while geometric genus measures the true topology of the curve after singularities are resolved.*

2.5.2 Singularities and Genus Reduction

Let the locus be defined by an implicit polynomial $F(x, y) = 0$.

Definition 4 (Singular Point)**.** *A point $P = (x_0, y_0)$ on the curve is singular if*

$$\nabla F(x_0, y_0) = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\right) = (0, 0).$$

Such points correspond to geometric features such as nodes, cusps, or self–intersections. Singularities arise when the parametrization fails to be locally injective or when multiple triangle configurations correspond to the same point in the plane.

2.5.3 Rational Curves and Lüroth's Theorem

With point $A = (1, 0)$ and point $B = (-1, 0)$ fixed, as the point C runs along the circumference of a unit circle, points specified in Figure 2 generate 1-dimensional traces. Since we have defined

$$X_n = (x(t), y(t))$$

where $t = \tan\left(\frac{\theta}{2}\right)$, then the locus is the image of $\mathbb{P}^1$, which is rational. Consequently, the Cartesian coordinates satisfy

$$\begin{cases} x = f_n(t) \\ y = g_n(t) \end{cases}$$

where $f_n, g_n \in \mathbb{R}(t)$. Hence, this elimination ensures that the genus is zero. Some triangle centers generate non-circular curves, but these (and some other triangle centers not plotted) arise because

- The parametrization overlaps;
- Or the curve crosses itself;

The theory of rational plane curves can be found in [1]. In particular, curves admitting rational parametrizations have genus zero. The degree–genus formula follows from the discussion in Fulton, *loc. cit.*

Theorem 7 (Lüroth's Theorem [2, Ch. 1], [4])**.** *Let K be a field and suppose*

$$K \subseteq L \subseteq K(t)$$

is an intermediate field between K and the rational function field $K(t)$. Then there exists a rational function $\phi(t)$ such that

$$L = K(\phi(t))$$

In particular, every intermediate field of $K(t)$ is itself a rational function field.

Corollary 5 (Rationality of Rationally Parametrized Curves)**.** *If a plane algebraic curve admits a rational parametrization*

$$x = f(t), \qquad y = g(t), \qquad f, g \in K(t),$$

then the function field of the curve is an intermediate field of $K(t)$.

Alternatively, by Lüroth's theorem [2, 4], the curve is rational and therefore has genus $g = 0$. Viewing this in the elimination theory context (see Subsection 2.4.1), we have established the following:

Theorem 8 (Euler Line Rationality Theorem)**.** *Let X_n be a triangle center lying on the Euler line of a right triangle inscribed in the unit circle. If the locus of X_n admits a rational parametrization*

$$(x(t), y(t)) \in \mathbb{R}(t)^2$$

Under the parameter $t = \tan(\theta/2)$, the locus is a rational curve of geometric genus 0.

Remark on tangent substitutions. The classical substitution $t = \tan\left(\frac{\theta}{2}\right)$ provides a rational parametrization of the unit circle,

$$(\cos\theta, \sin\theta) \mapsto \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$$

which establishes a birational correspondence between the circle and the projective line $\mathbb{P}^1$ (see [1, Ch. 1]). This substitution is widely used in algebraic geometry because it establishes a birational correspondence between the circle and the projective line. Consequently, loci derived from trigonometric parametrizations can be converted into rational algebraic curves. More generally, one may consider substitutions of the form

$$t = \tan\left(\frac{\theta}{2^n}\right), \qquad n \geq 1.$$

Such substitutions generate nested rational expressions for $\sin\theta$ and $\cos\theta$ through repeated application of the double-angle identities. Although the expressions become more complicated algebraically, they sometimes reduce the degree of the polynomials appearing during symbolic elimination procedures such as Gröbner basis computations.

Lemma 1 (Trigonometric Rationalization Lemma)**.** *Let θ be a real parameter and define*

$$t = \tan\left(\frac{\theta}{2^n}\right), \qquad n \geq 1.$$

Then $\sin\theta$ and $\cos\theta$ can be expressed as rational functions of t obtained through repeated application of the double-angle identities.

Proof. For $n = 1$, we have the classical identities

$$(\cos\theta, \sin\theta) \mapsto \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$$

If $t = \tan\left(\frac{\theta}{2^n}\right)$ with $n > 1$, then repeated use of the double-angle formulas

$$\cos(2\alpha) = \frac{1-\tan^2\alpha}{1+\tan^2\alpha}, \qquad \sin(2\alpha) = \frac{2\tan\alpha}{1+\tan^2\alpha}$$

expresses $\sin\theta$ and $\cos\theta$ as rational functions of t. $\square$

Computational remark. Although the substitutions $t = \tan(\theta/2^n)$ all lead to rational expressions, the classical choice $t = \tan(\theta/2)$ is usually the most efficient. It produces the lowest-degree rational expressions for $\sin\theta$ and $\cos\theta$, which significantly reduces the complexity of Gröbner basis computations when eliminating the parameter t.
We can also extend to the following theorem.

Theorem 9. *Let $\mathcal{C}$ denote the algebraic curve, describing the locus of a triangle center expressed in barycentric or trilinear coordinates as a rational function of the triangles angles $\angle A$, $\angle B$ and $\angle C$ under the constraint $\angle A + \angle B + \angle C = \pi$. Suppose the center lie on the*

Euler line so that its coordinates depend on a single angular parameter θ along the family of triangles, preserving the Euler-line alignment. After the classical half-angle substitution $t = \tan\left(\frac{\theta}{2}\right)$, all trigonometric expressions become rational functions of t, yielding a rational parametrization of $\mathcal{C}$. Consequently, $\mathcal{C}$ is birational to the projective line $\mathbb{P}^1$, and therefore the curve has genus zero.

This establishes that loci arising from Euler-line parametrizations admit rational parametrizations and belong to the class of genus-zero algebraic curves.

2.5.4 Real versus Complex Geometry

While the complex algebraic curve has genus zero and is therefore topologically equivalent to a sphere, the *real locus* observed in the Euclidean plane may exhibit diverse geometric structures depending on the analytic behavior of $\lambda_n(\theta)$.

Compact Loci. If $\lambda_n(\theta)$ remains finite on the interval $\theta \in [0, \pi]$, the locus forms a bounded curve such as a circle or ellipse.

Non–Compact Loci. If poles occur in $\lambda_n(\theta)$, branches of the curve may escape to infinity, producing non–compact real curves analogous to hyperbolas.

Connectivity. Although the complex curve is connected, the real locus $\mathcal{V}(\mathbb{R})$ may consist of several disjoint components depending on sign changes in the parametrization.

The results of Section 2 establish that loci arising from Euler-line centers in the Thales configuration admit rational parametrizations and therefore define algebraic curves of genus zero. Section 3 applies this theoretical framework to compute explicit loci of selected Kimberling centers and to illustrate the geometric behavior predicted by the algebraic analysis.

3 Explicit Loci of Kimberling Centers

This section applies the algebraic and geometric framework developed in Section 2 to compute and visualize loci of specific Kimberling centers in the Thales configuration. The goal is twofold: first, to demonstrate how the theoretical machinery of rational parametrization and elimination theory produces explicit algebraic curves; and second, to illustrate how these curves appear geometrically through computational visualization. Throughout this section, we adopt the coordinate normalization introduced earlier.

Let's revisit the Thales configuration of a unit circle centered at the origin O, where $A = (1,0)$, $B = (-1,0)$ and $C = (\cos\theta, \sin\theta)$, with $\theta \in [0,\pi)$. For triangle centers, lying on the Euler line $\overline{OC}$, the triangle center coordinates can be expressed as

$$X_n(\theta) = \lambda_n(\theta)(\cos\theta, \sin\theta)$$

where $\lambda_n(\theta)$ is the Euler scalar function associated with the center X_n. Above can be expressed parametrically as

$$\begin{cases} x(\theta) = \lambda_n(\theta)\cos\theta \\ y(\theta) = \lambda_n(\theta)\sin\theta \end{cases}$$

When λ_n is a constant, the locus is a circle. Otherwise, When λ_n varies with θ, the resulting curve may be a higher–degree rational variety whose implicit equation can be obtained by elimination.

3.1 Euler–Line Parametrization

In the Thales configuration the orthocenter lies on the circumcircle and therefore satisfies

$$H(\theta) = (\cos\theta, \sin\theta).$$

Any triangle center X_n lying on the Euler line may therefore be expressed as a scalar multiple of the orthocenter:

$$X_n(\theta) = \lambda_n(\theta)H(\theta) = \lambda_n(\theta)(\cos\theta, \sin\theta)$$

where $\lambda_n(\theta)$ is the associated Euler scalar function. Consequently the locus of X_n admits the polar parametrization

$$r(\theta) = |\lambda_n(\theta)|.$$

3.1.1 A Circularity Criterion

Proposition 1 (Circularity Criterion)**.** *Let X_n be a triangle center lying on the Euler line of a right triangle inscribed in a unit circle. The locus of X_n is a circle centered at the origin if and only if the Euler scalar function $\lambda_n(\theta)$ is a constant.*

Proof. If $\lambda_n(\theta) = k$ is a constant, then

$$X_n(\theta) = (k\cos\theta, k\sin\theta)$$

Eliminating θ yields

$$x^2 + y^2 = k^2$$

which is a circle of radius $|k|$ centered at the origin. Conversely, suppose the locus is a circle centered at the origin with radius R. Then every point on the locus satisfies

$$x^2 + y^2 = R^2.$$

Substituting the parametrization gives

$$\lambda_n(\theta)^2(\cos^2\theta + \sin^2\theta) = R^2,$$

Hence $\lambda_n(\theta)^2 = R^2$. Therefore $|\lambda_n(\theta)| = R$ is a constant, and the scalar function must be a constant. This completes the proof. $\square$

3.1.2 Computational Visualization

While symbolic elimination yields the implicit equations of the loci, geometric intuition is greatly aided by computational plotting. Dynamic geometry systems such as GeoGebra and Desmos allow the locus to be visualized as the vertex C moves along the circumcircle.

```
A = (1,0)
B = (-1,0)
C = (cos(t), sin(t))
Circle((0,0),1)
```

The triangle centers may then be generated using built–in commands or barycentric formulas, and the locus traced as t varies.

Similarly, Desmos may be used to plot parametric curves directly. For example, the centroid X_2 can be visualized, using

$$x(t) = \frac{1}{3}\cos t, \qquad y(t) = \frac{1}{3}\sin t.$$

The resulting trace immediately reveals that the locus is a circle of radius 1/3.

Such visualizations serve as experimental verification of the algebraic results derived in the following subsections.

3.1.3 Symbolic Computation of Loci

The loci of triangle centers are computed using the following steps:

1. Parametrize the moving vertex of the triangle by $D = (\cos\theta, \sin\theta)$.
2. Express the coordinates of the center X_n in terms of θ.
3. Apply the rational substitution $t = \tan\left(\frac{\theta}{2}\right)$ to convert trigonometric expressions into rational functions.
4. Eliminate the parameter t using Gröbner bases or resultants.
5. Compute the degree of the resulting algebraic curve.

Alternative substitutions of the form $t = \tan\left(\frac{\theta}{2^n}\right)$ may be used in special cases when they simplify the elimination step.

Step 1: Parametrize the triangle. The vertex C is represented by

$$C = (\cos\theta, \sin\theta).$$

Side lengths are expressed as

$$\begin{aligned} a &= \left|\overline{BC}\right| = 2\sin\left(\frac{\theta}{2}\right) \\ b &= \left|\overline{AC}\right| = 2\cos\left(\frac{\theta}{2}\right) \\ c &= \left|\overline{AB}\right| = 2 \end{aligned}$$

whereas angles are expressed as

$$\begin{aligned} \alpha =& \angle BAC = \frac{1}{2}\left(\pi - \theta\right) \\ \beta =& \angle ABC = \frac{\theta}{2} \\ \gamma =& \angle BCA = \frac{\pi}{2} \end{aligned}$$

Step 2: Express the center in barycentric/trilinear coordinates. Each Kimberling center possesses barycentric coordinates of the form $(\lambda_1 : \lambda_2 : \lambda_3)$. These are converted to Cartesian coordinates using

$$X_n(\theta) = \frac{\lambda_1 A + \lambda_2 B + \lambda_3 C}{\lambda_1 + \lambda_2 + \lambda_3} = \left(\frac{\lambda_1 - \lambda_2 + \lambda_3\cos\theta}{\lambda_1 + \lambda_2 + \lambda_3}, \frac{\lambda_3 \sin\theta}{\lambda_1 + \lambda_2 + \lambda_3}\right) \tag{2}$$

Alternatively, with three vertex coordinates A, B and C given, we can also utilize trilinear coordinates $(x : y : z)$ by writing

$$\vec{X_n} = \frac{ax}{ax+by+cz}\vec{A} + \frac{by}{ax+by+cz}\vec{B} + \frac{cz}{ax+by+cz}\vec{C} \tag{3}$$

where the side lengths are

$$\left|\vec{C}-\vec{B}\right| = a$$
$$\left|\vec{A}-\vec{C}\right| = b$$
$$\left|\vec{B}-\vec{A}\right| = c$$

Setting $\vec{A} = (1,0)$, $\vec{B} = (-1,0)$ and $\vec{C} = (\cos\theta, \sin\theta)$ yields the same result.

Step 3: Rationalize the parameter. Applying the half–angle substitution $t = \tan\left(\frac{\theta}{2}\right)$ transforms the trigonometric functions into rational expressions

$$(\cos\theta, \sin\theta) \mapsto \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$$

Thus the locus becomes a rational parametrization

$$X_n(\theta) = \left(\frac{P(t)}{R(t)}, \frac{Q(t)}{R(t)}\right)$$

Step 4: Eliminate the parameter. The implicit equation of the locus is obtained by eliminating t from the system

$$\begin{cases} xR(t) - P(t) = 0 \\ yR(t) - Q(t) = 0 \end{cases}$$

This elimination can be performed using resultants or Gröbner bases.

Step 5: Verify computationally. Numerical sampling of t allows the curve to be plotted and compared against the symbolic implicit equation. The following Python snippet illustrates the basic plotting approach.

```
import numpy as np
import matplotlib.pyplot as plt

theta = np.linspace(0,2*np.pi,1000)

x = (1/3)*np.cos(theta)
y = (1/3)*np.sin(theta)

plt.plot(x,y)
plt.gca().set_aspect('equal')
plt.show()
```

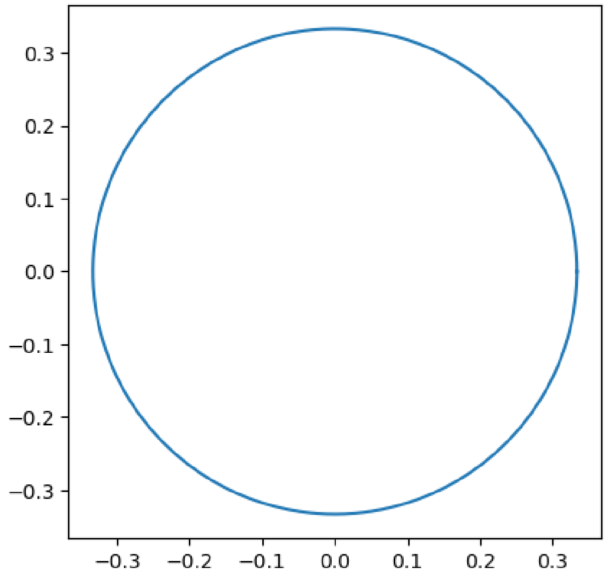

Figure 2: The locus of X_2

This produces the circular locus of the centroid.
The following script verifies this property symbolically for the centroid X_2, whose Euler scalar is a constant.

```
import sympy as sp

parameter

t = sp.symbols('t', real=True)

half-angle substitution

cos_theta = (1 - t**2)/(1 + t**2)
sin_theta = (2*t)/(1 + t**2)

centroid (Euler scalar lambda = 1/3)

lam = sp.Rational(1,3)

x = sp.simplify(lam * cos_theta)
y = sp.simplify(lam * sin_theta)

print("x(t) =", x)
print("y(t) =", y)

confirm rational structure
```

```
num_x, den_x = sp.together(x).as_numer_denom()
num_y, den_y = sp.together(y).as_numer_denom()

print("Numerator of x:", sp.factor(num_x))
print("Denominator of x:", sp.factor(den_x))

print("Numerator of y:", sp.factor(num_y))
print("Denominator of y:", sp.factor(den_y))
```

which yields

```
x(t) = (1 - t**2)/(3*(t**2 + 1))
y(t) = 2*t/(3*(t**2 + 1))
Numerator of x: -(t - 1)*(t + 1)
Denominator of x: 3*(t**2 + 1)
Numerator of y: 2*t
Denominator of y: 3*(t**2 + 1)
```

The output confirms that both coordinates are ratios of polynomials in t. Hence the locus admits a rational parametrization discovered earlier. This computational verification aligns with the algebraic conclusions of Section 2: loci arising from Euler-line centers in the present configuration are rational curves and therefore have geometric genus zero.

For more complicated centers, the functions $x(\theta)$ and $y(\theta)$ can be substituted directly into the plotting routine, allowing the algebraic predictions of Section 2 to be explored experimentally.

These computational tools therefore complement the algebraic derivations, providing geometric intuition for the loci studied throughout the remainder of this work.

3.2 Invariant and Variant Euler-Line Centers

The algebraic nature of the loci derived in the previous subsection suggests a natural dichotomy between centers whose Euler-line parameter is a constant and those whose parameter varies with the shape of the triangle. This distinction determines whether the resulting locus is circular or a more general rational curve.

Definition 5 (Euler-Line Parameter)**.** *Let $\triangle ABC$ be a right triangle inscribed in the unit circle and let X_n be a triangle center lying on the Euler line. The* Euler-line parameter $\lambda_n(\theta)$ *is defined by*

$$X_n = O + \lambda_n(\theta)\,(H - O),$$

where O is the circumcenter, H is the orthocenter, and θ denotes the shape parameter of the right triangle on the unit circle.

Definition 6 (Shape Invariance)**.** *A triangle center X_n is said to be* shape-invariant *on the Euler line if its parameter $\lambda_n(\theta)$ is a constant for all admissible θ. Otherwise, it is called* shape-variant.

Theorem 10 (Euler-Line Rationality Theorem)**.** *Let X_n be a triangle center lying on the Euler line of a right triangle $\triangle ABC$ inscribed in the unit circle. Then the locus of X_n as the triangle varies is a rational algebraic curve of genus zero. Moreover,*

1. *If $\lambda_n(\theta)$ is a constant, the locus is a circle centered at the circumcenter.*
2. *If $\lambda_n(\theta)$ is nonconstant but rational in θ, the locus is a rational plane curve birational to $\mathbb{P}^1$.*

Proof. Let $\triangle ABC$ be a right triangle inscribed in the unit circle with vertices parameterized by a single angular parameter θ. Because the circumcircle is fixed, the circumcenter O is a constant and the orthocenter H is a rational function of the triangle's coordinates.

Since X_n lies on the Euler line, it admits the affine representation:

$$X_n = O + \lambda_n(\theta)(H - O).$$

The coordinates of H are rational functions of the vertex coordinates, and those coordinates themselves are trigonometric functions of θ. After the standard substitution $t = \tan\left(\frac{\theta}{2}\right)$, all coordinates become rational functions in t. Consequently the coordinates of X_n are rational in t.

Thus the locus admits a rational parametrization in the form $(x(t), y(t))$, which defines a rational plane curve. By a classical result of algebraic geometry (cf. Lüroth's theorem), any intermediate field between $\mathbb{C}$ and $\mathbb{C}(t)$ is rational, implying that the function field of the locus is isomorphic to $\mathbb{C}(t)$. Therefore the curve has genus zero.

If λ_n is a constant, then X_n is obtained by a fixed affine combination of O and H, so its distance from O remains constant as θ varies. Hence the locus is a circle. If λ_n varies rationally, the resulting curve remains rational but is generally non-circular. □

The theorem shows that Euler-line centers of right triangles naturally produce rational curves. This explains the prevalence of circles, cardioid-like curves, and other genus-zero loci in computational experiments using dynamic geometry software such as GeoGebra or Desmos.

Choice of rational parameter. In practical computations the substitution $t = \tan\left(\frac{\theta}{2}\right)$ is particularly convenient because it converts trigonometric expressions into rational functions at an early stage of the elimination process. This significantly improves the performance of symbolic tools such as Gröbner basis algorithms.

In some cases it is advantageous to use higher-order substitutions

$$t = \tan\left(\frac{\theta}{2^n}\right), \qquad n \geq 2,$$

which generate rational expressions through successive double-angle relations. While these substitutions may introduce larger intermediate expressions, they can occasionally reduce the degree of the resulting algebraic curve after elimination. Such techniques are therefore useful when exploring loci computationally.

Example. The locus of the centroid whose triangle center is X_2 can be presented as

$$X_2(\theta) = \left(\frac{1}{3}\cos\theta, \frac{1}{3}\sin\theta\right)$$

where $\lambda_2(\theta) = \frac{1}{3}$. The expression is equivalent to

$$x^2 + y^2 = \frac{1}{9}$$

These results fundamentally follow that X_2 is located at exactly 1/3 of the altitude length from the base, hence the barycentric coordinates $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

Alternatively, if the trilinear coordinates are

$$\left(\frac{1}{a} : \frac{1}{b} : \frac{1}{c}\right) \equiv \left(\frac{1}{2\sin\frac{\theta}{2}} : \frac{1}{2\cos\frac{\theta}{2}} : \frac{1}{2}\right) \equiv \left(\csc\frac{\theta}{2} : \sec\frac{\theta}{2} : 1\right)$$

then coefficients in Equation 3 collapse to

$$\frac{ax}{ax+by+cz} = \frac{by}{ax+by+cz} = \frac{cz}{ax+by+cz} = \frac{1}{3}$$

which yield the same curve.

Example. The locus of the nine-point center whose triangle center is X_5 can be presented as

$$X_5(\theta) = \left(\frac{1}{2}\cos\theta, \frac{1}{2}\sin\theta\right)$$

where $\lambda_5(\theta) = \frac{1}{2}$. The expression is equivalent to

$$x^2 + y^2 = \frac{1}{4}$$

The geometric property shows that the nine-point center lies on the Euler line of $\triangle ABC$, at the midpoint between the orthocenter X_4 and the circumcenter X_3.

Example. Following Section 2.2, the locus of the circumcenter whose triangle center is X_3 collapses to a single point. This example shows the degenerate circular case, where $\lambda_3(\theta) = 0$.

3.3 Shape-Variant Centers

We now consider Euler-line centers whose Euler scalar function $\lambda_n(\theta)$ varies with the triangle's shape. In this situation the distance of the center from the circumcenter is no longer fixed, and the resulting locus becomes a non-circular rational curve. Recall from Section 3.1 that every Euler-line center admits the parametrization

$$X_n(\theta) = \lambda_n(\theta)(\cos\theta, \sin\theta).$$

When $\lambda_n(\theta)$ depends on θ, the locus becomes the polar curve

$$r(\theta) = |\lambda_n(\theta)|$$

Consequently the Cartesian parametrization takes the form

$$\begin{cases} x(\theta) = \lambda_n(\theta)\cos\theta \\ y(\theta) = \lambda_n(\theta)\sin\theta \end{cases}$$

If the scalar function $\lambda_n(\theta)$ is rational in the half–angle parameter $t = \tan\left(\frac{\theta}{2}\right)$, then after the half-angle substitution both coordinates become rational functions of t. Therefore the locus admits a rational parametrization

$$X_n(t) = \left(\frac{P(t)}{R(t)}, \frac{Q(t)}{R(t)}\right)$$

where $P(t)$, $Q(t)$ and $R(t)$ are polynomials. It follows that the resulting algebraic curve is rational and hence has genus zero.

Example. Consider a center X_n whose Euler scalar has the form

$$\lambda_n(\theta) = \frac{1}{3+\cos\theta}.$$

The parametrization becomes

$$x(\theta) = \frac{\cos\theta}{3+\cos\theta}, \qquad y(\theta) = \frac{\sin\theta}{3+\cos\theta}.$$

Applying the half–angle substitution yields rational functions of the parameter t, and elimination of t produces an algebraic curve of finite degree. Such loci typically appear as quartic or higher-degree rational curves.

Geometric interpretation. For shape-variant centers the point X_n moves along the Euler line with a parameter-dependent scaling factor. As the orthocenter traces the circumcircle, the varying scalar stretches and compresses the radial distance, producing a noncircular algebraic locus. These loci form the primary examples studied in the subsequent computations.

Proposition 2. *If $\lambda_n(\theta)$ is a rational function of $\sin\left(\frac{\theta}{2}\right)$ and $\cos\left(\frac{\theta}{2}\right)$, then the locus of X_n admits a rational parametrization after the substitution $t = \tan\left(\frac{\theta}{2}\right)$.*

Proof. Substituting $t = \tan\left(\frac{\theta}{2}\right)$ expresses $\sin\theta$ and $\cos\theta$ as rational functions of t. Since $\lambda_n(\theta)$ is rational in these quantities, both coordinates

$$\begin{cases} x(\theta) = \lambda_n(\theta)\cos\theta \\ y(\theta) = \lambda_n(\theta)\sin\theta \end{cases}$$

become rational functions of t. Therefore, the locus admits a rational parametrization and defines a rational algebraic curve. □

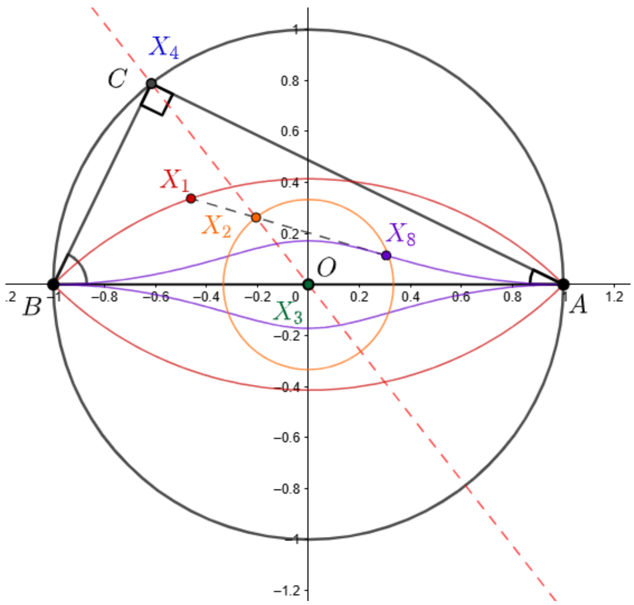

Figure 3: The locus of Nagel point X_8 in purple. Note that X_8, X_1 (incenter) and X_2 are collinear.

Example. In the Thales configuration, trilinear coordinates of a Nagel point whose center is X_8 are

$$\left(\csc^2\left(\frac{\alpha}{2}\right) : \csc^2\left(\frac{\beta}{2}\right) : \csc^2\left(\frac{\gamma}{2}\right)\right) = \left(\csc^2\left(\frac{1}{4}(\pi-\theta)\right) : \csc^2\left(\frac{\theta}{4}\right) : 2\right)$$

and its barycentric coordinates are

$$(s-a : s-b : s-c)$$

where $s = \frac{1}{2}(a+b+c)$ denotes the semiperimeter of $\triangle ABC$. Comparing the angles of $\triangle ABC$, trilinear coordinates and barycentric coordinates, we can consider barycentric coordinates since the "deepest" angle is $\frac{1}{2}$. Using the substitution $t = \tan\left(\frac{\theta}{4}\right)$ for Equation 2, where $2 \cdot 2 = 4$ trigonometric substitutions are

$$\cos\left(\frac{\theta}{2}\right) = \frac{1-t^2}{1+t^2} \qquad \sin\left(\frac{\theta}{2}\right) = \frac{2t}{1+t^2}$$
$$\cos\theta = \frac{1-6t^2+t^4}{(1+t^2)^2} \qquad \sin\theta = \frac{4t(1-t^2)}{(1+t^2)^2}$$

The explicit form of $X_8(\theta)$ is

$$X_8(t) = \left(\frac{(1-2t+3t^2)(1+2t-3t^2)}{(1+t^2)^2}, \frac{4t^2(1-t)^2}{(1+t^2)^2}\right)$$

with $t \in [0,1]$. This result is consistent with the geometric property that the coordinate position of X_8 lies on the *Nagel line*, where

$$X_8 + 2X_1 = 3X_2$$

with X_1 denoted as the incenter and X_2 denoted as the centroid. The implicit form of the curve in the upper plane, where $F(x,y) = 0$, is

$$x^4 + 2x^2y^2 - 2x^2 + y^4 - 8y^3 + 14y^2 - 8y + 1 = 0$$

On the other hand, if we consider trilinear coordinates, then the substitution to use is $t = \tan\left(\frac{\theta}{8}\right)$, where we work with $2 \cdot 3 = 6$ unique cos and sin substitutions:

$$\cos\left(\frac{\theta}{4}\right) = \frac{1-t^2}{1+t^2} \qquad \sin\left(\frac{\theta}{4}\right) = \frac{2t}{1+t^2}$$
$$\cos\left(\frac{\theta}{2}\right) = \frac{1-6t^2+t^4}{(1+t^2)^2} \qquad \sin\left(\frac{\theta}{2}\right) = \frac{4t(1-t^2)}{(1+t^2)^2}$$
$$\cos\theta = \frac{1-28t^2+70t^4-28t^6+t^8}{(1+t^2)^4} \qquad \sin\theta = \frac{8t(1-t^2)(1-6t^2+t^4)}{(1+t^2)^4}$$

These trilinear substitutions also reveal the rational curve with coordinates whose degrees in the numerator and the denominator are greater than the barycentric substitutions, which simplify $X_8(\theta)$ significantly.

Example. In the Thales configuration, trilinear coordinates of a Feuerbach point whose center is X_{11} are

$$\begin{aligned}&(1-\cos(\beta-\gamma) : 1-\cos(\gamma-\alpha) : 1-\cos(\alpha-\beta))\\ &= \left(1-\sin\left(\frac{\theta}{2}\right) : 2\sin^2\left(\frac{\theta}{4}\right) : 1-\sin\theta\right)\end{aligned}$$

and its barycentric coordinates are

$$\left((s-a)(b-c)^2 : (s-b)(c-a)^2 : (s-c)(a-b)^2\right)$$

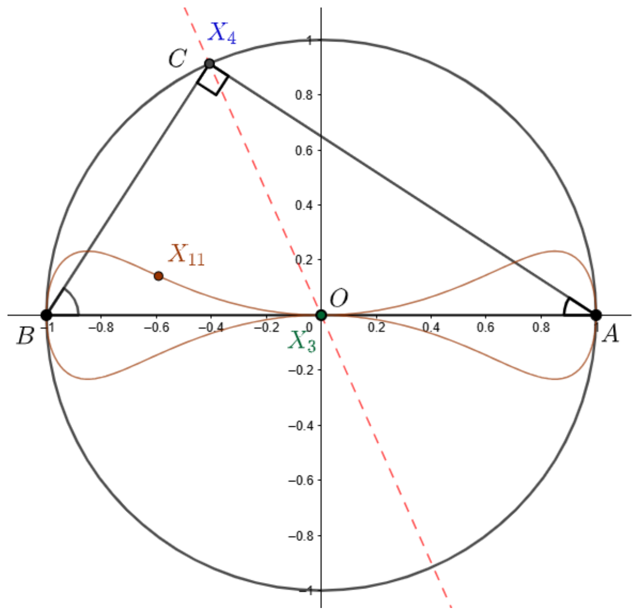

Figure 4: The locus of X_{11}

where $s = \frac{1}{2}(a+b+c)$ denotes the semiperimeter. As the squared exponent does not affect the "deepest" angle $\frac{1}{2}$, the substitution $t = \tan\left(\frac{\theta}{4}\right)$ can be applied in the case of barycentric coordinates. The parametric curve is

$$X_{11}(t) = \left(\frac{1 - 4t + 3t^2 + 8t^3 - 13t^4 - 4t^5 + t^6}{(1+t^2)^2(1-4t+5t^2)}, \frac{2t(1-t)(1-2t-t^2)^2}{(1+t^2)^2(1-4t+5t^2)} \right)$$

with $t \in [0,1]$. The implicit form for the curve in the upper plane is

$$(x^2+y^2)^3 - (1+2y)(x^2+y^2)^2 + 2y(x^2+y^2) + y^2 = 0$$

These examples illustrate that even geometrically intricate loci arising from triangle centers remain algebraically simple: despite their complex appearance, the curves belong to the class of rational plane curves and therefore possess genus zero.

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