Classes of Logarithmic Fractional Part Function

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Abstract

This paper investigates the analytical properties and evaluation of a class of integrals involving the fractional part of logarithmic arguments, $\int_0^1 \cdots \int_0^1 \{\ln(\prod x_i)\}^k \prod dx_i$. Originating from recent computational challenges in the online mathematical community, these expressions are shown to reveal deep structural connections to Stieltjes-type constants and higher-order logarithmic moments. We formalize the evaluation process by introducing a kernel formulation, $Q(x)$, and utilizing Feynman's parameterization trick—specifically differentiation under the integral sign—to reduce multivariable product structures into univariate baseline integrals.A central focus of this work is the comparative analysis of fractional part conventions for negative arguments. We contrast the standard number-theoretic definition $\{x\} = x - \lfloor x \rfloor$ with the Knuth (symmetric) convention, demonstrating that the choice of operator fundamentally reconfigures the resulting series expansions and the convergence behavior of their coefficients. Through the application of Stirling numbers of the second kind and periodic decomposition, we derive closed-form representations for these integrals and establish criteria for the admissibility of their extensions. The results provide a template for analyzing non-periodic kernels and offer a benchmark for the limits of analytic continuity in logarithmic fractional systems.

1 Motivation

Integrals involving the fractional part function arise in many areas of mathematics, including analytic number theory, probability theory, and harmonic analysis[6, 5]. While the function itself is elementary, its interaction with analytic kernels often reveals surprisingly deep structures. In particular, integrals involving logarithmic or reciprocal transformations frequently introduce oscillatory behaviors governed by the integer–part function.

A recent set of integrals shared on X / Twitter involved expressions such as

$$\int_0^1 \int_0^1 \int_0^1 \{\ln(xyz)\} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

as well as related variants involving powers of xyz or higher powers of the fractional part. These integrals are defined entirely on the unit cube and therefore appear at first to be simple Euler-type integrals. However, the presence of the logarithm introduces a key phenomenon: for $0 < x < 1$, we have

$$\ln(x) < 0$$

Consequently, the integrand involves fractional parts of negative numbers, a situation where the definition of the fractional part function is not universally agreed upon. Two conventions are commonly used in the literature[4, 3]. The definition favored in number theory defines the fractional part as

$$\{x\} = x - \lfloor x \rfloor$$

while an alternative symmetric definition frequently used in Fourier analysis extends the function by

$$\{x\} = \begin{cases} x - \lfloor x \rfloor, & x \geq 0 \\ x - \lceil x \rceil, & x < 0 \end{cases}$$

Both conventions coincide for positive arguments but differ for negative ones. As a result, integrals involving logarithmic arguments may yield distinct analytic forms depending on the adopted convention. The phenomenon becomes particularly interesting when such integrals are embedded into higher–dimensional domains. For example, integrals over the n-dimensional unit cube involving products of variables,

$$\int_{(0,1)^n} f\left(\prod_{i=1}^{n} x_i\right) \, \mathrm{d}x_1 \ldots \mathrm{d}x_n$$

naturally arise in probabilistic models, where the variables are independent and uniformly distributed. In this setting, logarithmic transformations convert multiplicative structures into additive ones, leading to Mellin-type kernels and generating functions connected to classical special functions.

One representative family of integrals considered in this article is

$$I_L(\alpha,\beta,n)=\int\limits_{(0,1)^n}\left(\prod_{i=1}^{n}x_i\right)^{\alpha-1}\left\{\ln\left(\prod_{i=1}^{n}x_i\right)\right\}\mathrm{d}x_1\ldots\mathrm{d}x_n$$

where $\Re(\alpha)>0$, and where $\Re(\beta)>0$ is a positive integer. integrals of this type reveal connections between fractional part functions, Mellin transforms, and probability distributions associated with products of uniform random variables. The purpose of this paper is to investigate **classes of integrals involving fractional parts of logarithmic expressions**, with particular emphasis on how their analytic structures depend on the chosen fractional-part convention. We demonstrate that these integrals naturally lead to Mellin kernels, periodic decompositions, and expansions, involving special functions like Hurwitz-zeta function and Stieltjes constants[1, 6].

The central question guiding this work is therefore the following:

> Can analytic integrals detect the difference between competing definitions of the fractional part function?

We show that the answer is affirmative. When logarithmic transformations produce negative arguments, the two conventions lead to distinct analytic decompositions and ultimately to different representations in terms of special functions. The remainder of the paper develops this phenomenon systematically. After reviewing the definitions of the fractional part function, we introduce probability reductions that convert multidimensional integrals into Mellin-type kernels, analyze the periodic structure of the resulting integrals, and derive explicit formulas under both conventions.

2 Fractional Part Definitions

The fractional part function plays a central role in the integrals studied in this paper. While the function is elementary in appearance, subtle issues arise when extending it to negative arguments. Because many of the integrals considered here involve logarithmic expressions over the unit interval, negative arguments occur naturally, making the choice of definition essential.

2.1 Classical Definition

The most widely used definition of the fractional part function is

$$\{x\} = x - \lfloor x \rfloor$$

where $\lfloor x \rfloor$ denotes the floor function, the greatest integer less than or equal to x[3]. Under this definition, the function satisfies

$$0 \leq \{x\} < 1$$

for all real values x. The function is periodic with period 1, which is mathematically expressed as

$$\{x + 1\} = \{x\},$$

and exhibits jump discontinuities at every integer. For positive real values, this definition behaves intuitively. For example,

$$\{2.37\} = 0.37$$

However, for negative numbers, the definition yields values near 1. For instance,

$$\{-0.2\} = 0.8$$

Thus, the classical definition produces a **sawtooth waveform shifted upward** for negative arguments.

2.2 Symmetric Definition

An alternative extension sometimes used in harmonic analysis and signal theory[4] modifies the definition for negative arguments:

$$\{x\} = \begin{cases} x - \lfloor x \rfloor, & x \geq 0 \\ x - \lceil x \rceil, & x < 0 \end{cases}$$

where $\lceil x \rceil$ denotes the ceiling function, the least integer greater than or equal to x. Under this convention, the fractional part remains symmetric around zero in the sense that

$$\{-x\} = -\{x\}$$

This version behaves more like an odd periodic function and produces values in the interval

$$-1 < \{x\} < 1$$

Although both definitions coincide for $x \geq 0$, they differ substantially for negative inputs. Because logarithmic transformations frequently produce negative values, this distinction becomes analytically significant in the integrals studied here.

2.3 Behavior Under Logarithmic Transformations

A key observation motivating the present work is that

$$0 < x < 1 \qquad \Rightarrow \qquad \ln(x) < 0$$

Thus, even though many of our integrals are taken over the interval $(0, 1)$, the fractional part function is applied to **negative quantities**, such as

$$\{\ln(x)\}, \qquad \{\ln(xy)\}, \qquad \{\ln(xyz)\}$$

Consequently, the value of the integrand depends on which convention for the fractional part function is adopted. This situation leads to a surprising phenomenon: two integrals defined over the **same domain** may produce **different analytic evaluations** solely because of the underlying definition of the fractional part function.

2.4 Periodic Structure

Regardless of the chosen convention, the fractional part function remains periodic with period 1:

$$\{x + 1\} = \{x\}.$$

Therefore it admits a representation as a periodic function with jump discontinuities at the integers. This property makes the function particularly suitable for decomposition using Fourier series or other periodic expansions. Later sections will exploit this periodic structure to evaluate integrals of the form

$$\int_0^1 K(x)\,\{\ln(x)\}\,\mathrm{d}x$$

and related multidimensional generalizations.

2.5 Implications for Integration

The two definitions leads to different behaviors when integrated against analytic kernels. In particular:

- Mellin-type kernels may produce different analytic continuations

- Fourier expansions differ by constant shifts
- Certain integrals acquire additional correction terms

These differences provide a natural framework for investigating whether analytic integrals can **detect the underlying fractional-part convention**. The remainder of this paper develops this idea systematically. In the next section, we introduce a probabilistic viewpoint that allows many multidimensional integrals over the unit cube to be reduced to simpler one-dimensional forms.

3 Uniform Product Reduction and Logarithmic Moments

This section establishes the probabilistic and analytic framework used throughout the paper. Rather than beginning with one-dimensional logarithmic integrals, we start from a product of independent uniform variables. This approach naturally generates the logarithmic powers appearing in later sections and leads to a reduction to a one-dimensional integral representation.

3.1 Independent Uniform Variables

Let $X_1, X_2, \cdots, X_n$ denote independent random variables distributed uniformly on the unit interval $X_i - \sim U(0,1)$. The joint density on the unit cube $(0,1)^n$ is therefore

$$f(x_1, x_2, \ldots, x_n) = 1$$

Consequently, for any integrable function F,

$$\mathbb{E}[F(X_1, \ldots, X_n)] = \int\limits_{(0,1)^n} F(x_1, \cdots, x_n) \, \mathrm{d}x_1 \ldots \mathrm{d}x_n$$

This representation allows multidimensional integrals over the unit cube to be interpreted probabilistically.

3.2 Product Variable

Define the product

$$Y = \prod_{i=1}^{n} X_i.$$

Since each X_i lies in the unit interval, the product satisfies $0 < Y < 1$. Many integrals arising in the present work depend only on this product variable.

Property 3.2.1. *If F depends only on the product Y, then*

$$\int\limits_{(0,1)^n} F\left(\prod_{i=1}^{n} x_i\right) \mathrm{d}\mathbf{x}$$

reduces to an integral, involving the single variable Y.

The mechanism for this reduction is obtained through a logarithmic transformation.

3.3 Logarithmic Transformation

Define $t_i = -\ln(x_i)$, where $i \in \{1, \ldots, n\}$. Since $0 < x_i < 1$, each transformed variable satisfies $t_i > 0$. The inverse transformation is

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} e^{-t_1} \\ e^{-t_2} \\ \vdots \\ e^{-t_n} \end{bmatrix}$$

Thus, since the Jacobian determinant is

$$\left| \det \begin{bmatrix} -e^{-t_1} & 0 & \cdots & 0 \\ 0 & -e^{-t_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & -e^{-t_n} \end{bmatrix} \right| = e^{-(t_1+t_2+\cdots+t_n)}$$

Then

$$\mathrm{d}x_1 \ldots \mathrm{d}x_n = e^{-(t_1+t_2+\cdots+t_n)} \,\mathrm{d}t_1 \ldots \mathrm{d}t_n$$

Property 3.3.1. *Under this transformation, the product variable becomes*

$$Y = e^{-(t_1+t_2+\cdots+t_n)}$$

Taking logarithm gives

$$-\ln(Y) = t_1 + \cdots + t_n$$

Thus, the logarithm of the product corresponds to the sum of the transformed variables.

3.4 Simplex Reduction

Let $s = t_1 + \cdots + t_n$. The region $t_i > 0$ with fixed sum s forms an $(n-1)$-dimensional simplex. The volume of this simplex is

$$\frac{s^{n-1}}{(n-1)!}$$

This geometric factor is responsible for the logarithmic powers that appear in the reduced integral.

Lemma 3.4.1 (Simplex Integral)**.** *For integrable functions G,*

$$\int_{\sum_{i=1}^{n} t_i = s} \mathrm{d}t_1 \cdots \mathrm{d}t_n = \frac{s^{n-1}}{(n-1)!}$$

3.5 Product-Uniform Reduction

Combining the previous results yields the following reduction formula:

Theorem 3.5.1 (Product-Uniform Reduction)**.** *Let F be integrable on* $(0,1)$. *Then,*

$$\int_{(0,1)^n} F\left(\prod_{i=1}^{n} x_i\right) \mathrm{d}x_1 \cdots \mathrm{d}x_n = \frac{(-1)^{n-1}}{(n-1)!}\int_0^1 F(x)\ln^{n-1}(x)\,\mathrm{d}x$$

Proof. Using the transformation $x_i = e^{-t_i}$, the product becomes $x = e^{-s}$, where $s = t_1 + \cdots + t_n$. Integrating over the simplex produces the factor

$$\frac{s^{n-1}}{(n-1)!}$$

Substituting $s = -\ln(x)$ yields $(-\ln(x))^{n-1}$, which proves the theorem after the term arrangement. □

3.6 Fractional Logarithmic Integrals

Applying that theorem to functions, depending on the logarithm of the product, yields the integral form that will be studied in the remainder of the paper. Let

$$F(x) = \{\ln(x)\}^{\beta}\, x^{\alpha-1}$$

Then

$$\int_{(0,1)^n} \left\{\ln\left(\prod_{i=1}^{n} x_i\right)\right\}^{\beta} \left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} = \frac{(-1)^{n-1}}{\Gamma(n)}\int_0^1 \{\ln(x)\}^{\beta}\, x^{\alpha-1}\ln^{n-1}(x)\,\mathrm{d}x$$

where $\Gamma(n) = (n-1)!$ denotes the Gamma function, which is related to the factorial.

3.7 Remarks

1. The logarithmic power $\ln^{n-1}(x)$ allows multidimensional integrals, involving products of uniform variables to be studied through one-dimensional logarithmic integrals.

2. The factor $\Gamma(n) = (n-1)!$ appears naturally from the simplex volume.

3. This reduction allows multidimensional integrals, involving products of uniform variables to be studied through one-dimensional logarithmic integrals.

4 Fractional Logarithmic Integrals

Section 3 showed that integrals, involving products of independent random variables, can be reduced to one-dimensional logarithmic integrals. We now introduce the principal integral class that will be studied throughout the remainder of the paper.

A central feature of these integrals is that logarithmic powers arise naturally through differentiation with respect to a parameter. This observation allows families of integrals to be generated systematically, using the **Feynman's trick**, or differentiation under the integral sign.

4.1 Base Fractional Logarithmic Integral

Consider the following parametrized integral

$$J(\alpha,\beta) = \int_0^1 x^{\alpha-1}\left\{\ln(x)\right\}^{\beta}\,\mathrm{d}x$$

where $\Re(\alpha) > 0$ and β is a non-negative integer. This integral forms the analytic foundation for the classes of integrals studied in later sections.

4.2 Logarithmic Power Generation

Logarithmic powers may be introduced by differentiation with respect to the parameter α.

Lemma 4.2.1. *For $0 < x < 1$, $\frac{\partial}{\partial\alpha}x^{\alpha} = x^{\alpha}\ln(x)$. Generally, if m is an integer, then*

$$\frac{\partial^m}{\partial\alpha^m}x^{\alpha} = x^{\alpha}\ln^m(x)$$

4.3 Feynman Differentiation Principle

Applying differentiation under the integral sign[2] generates a family of logarithmic integrals.

Theorem 4.3.1. *Let*

$$J(\alpha,\beta) = \int_0^1 x^{\alpha-1}\left\{\ln(x)\right\}^{\beta}\,\mathrm{d}x$$

Then, for integers $n \geq 1$,

$$\frac{\partial^{n-1}}{\partial\alpha^{n-1}}J(\alpha,\beta) = x^{\alpha-1}\left\{\ln(x)\right\}^{\beta}\ln^{n-1}(x)\,\mathrm{d}x$$

Corollary 4.3.1. *Define*

$$I(\alpha,\beta,n)=\frac{(-1)^{n-1}}{\Gamma(n)}\int_0^1 \{\ln(x)\}^{\beta}\,x^{\alpha-1}\ln^{n-1}(x)\,\mathrm{d}x$$

Then

$$I(\alpha,\beta,n)=\frac{(-1)^{n-1}}{\Gamma(n)}\frac{\partial^{n-1}}{\partial\alpha^{n-1}}J(\alpha,\beta)$$

Thus, the entire family $I(\alpha,\beta,n)$ can be generated by differentiating the base integral $J(\alpha,\beta)$.

4.4 Structural Decomposition

The integrals under consideration have the general form

$$\int_0^1 f(x)g(x)\,\mathrm{d}x$$

where $f(x)=\{\ln(x)\}^{\beta}$ contains the fractional logarithmic structure and $g(x)=x^{\alpha-1}\ln^{n-1}(x)$ forms an analytic kernel generated through the parameter differentiation. This separation between a **fractional component** and an **analytic kernel** will be exploited in later sections.

4.5 Logarithmic Interval Geometry

Because $0<x<1$, the logarithm satisfies $\ln(x)<0$. Consequently, the argument of the fractional part function lies on the negative real axis. The behavior of $\{\ln(x)\}$ therefore depends on how the interval $(0,1)$ is partitioned by logarithmic scale.

Lemma 4.5.1 (Logarithmic Partition)**.** *For integers $k\geq 0$, $k\leq -\ln(x)<k+1$ if and only if $e^{-(k+1)}<x\leq k$.*

Corollary 4.5.1. *The interval* $(0,1)$ *admits the decomposition*

$$(0,1)=\bigcup_{k=0}^{\infty}\left(e^{-(k+1)},e^{-k}\right]$$

Thus, integrals involving $\{\ln(x)\}$ may be decomposed into an infinite sequence of logarithmically scaled intervals.

4.6 Outlook

The representation above highlights two complementary analytic approaches for evaluating fractional logarithmic integrals:

1. **Periodic methods**, based on the periodic structure of the fractional part function.

2. **Algebraic methods**, based on polynomial expansions within each logarithmic interval.

These approaches will be developed in the later sections of this paper.

5 Kernel Formulation of Fractional Logarithmic Integrals

The integrals introduced in the previous section possess a natural factorization into two components: a fractional logarithmic factor and an analytic kernel. This decomposition allows broader classes of integrals to be generated while preserving the essential analytic structure.

5.1 Kernel Decomposition

Consider integrals of the form

$$I = \int_0^1 f(x)g(x)\,\mathrm{d}x$$

where $f(x) = \{\ln(x)\}^{\beta}$ contains the fractional part structure, and $g(x)$ is an analytic function on the interval $(0,1)$. In the specific case studied in Section 4,

$$g(x) = x^{\alpha-1}\ln^{n-1}(x)$$

This kernel arises naturally through differentiation with respect to the parameter α.

5.2 Canonical Kernel

Motivated by the reduction obtained in Section 3, we define the canonical kernel

$$P_{\alpha,n}(x) = \frac{(-1)^{n-1}}{\Gamma(n)}x^{\alpha-1}\ln^{n-1}(x)$$

The principal integral of this paper may therefore be written as

$$I(\alpha,\beta,n) = \int_0^1 \{\ln(x)\}^{\beta}\,P_{\alpha,n}(x)\,\mathrm{d}x$$

This representation separates the **fractional logarithmic structure** from the **analytic kernel**.

5.3 Kernel Class

The formulation introduced above can be generalized further. The additional factor $Q(x)$ need not be analytic. Instead, it may contain algebraic, logarithmic, fractional structures, or even mixed structures, provided that the resulting integral remains convergent. Accordingly, we consider integrals of the form

$$I_Q = \int_0^1 \{\ln(x)\}^\beta P_{\alpha,n}(x) Q(x)\,\mathrm{d}x$$

Examples include x^μ, $a^{\ln(x)}$, $\{\ln(x)\}^\gamma$, or even more exotic expressions $e^{\{\ln(x)\}}$. Because such factors may combine analytic and fractional behavior, a simple classification of kernels is not possible. Instead, admissibility is determined by structural compatibility with the logarithmic partition of the unit interval.

Property 5.3.1 (Logarithmic Compatibility)**.** *Let*

$$I_Q = \int_0^1 \{\ln(x)\}^\beta P_{\alpha,n}(x) Q(x)\,\mathrm{d}x.$$

Suppose that $Q(x)$ is measurable on $(0,1)$, and the integral is absolutely convergent. Then, the integral may be decomposed as

$$I_Q = \sum_{k=0}^{\infty} \int_{e^{-(k+1)}}^{e^{-k}} \{\ln(x)\}^\beta P_{a,n}(x) Q(x)\,\mathrm{d}x.$$

Lemma 5.3.1 (Local Representation)**.** *On each of logarithmic intervals $e^{-(k+1)} < x \leq e^{-k}$, define $y = -\ln(x) - k$, where $0 \leq y < 1$. Then, for each integer $k \geq 0$, the integral is*

$$\int_0^1 \left\{\ln\left(e^{-(k+y)}\right)\right\}^\beta P_{\alpha,n}\left(e^{-(k+y)}\right) Q\left(e^{-(k+y)}\right) e^{-(k+y)}\,\mathrm{d}y$$

Remark. This lemma simply follows the substitution $x = e^{-(k+y)}$. As noted in Section 2, taking the fractional part of a negative real value depends on the choice of the adopted convention when evaluating the integral. No simplification is imposed at this stage. The explicit evaluation of the fractional part function will be deferred to later sections, where different representations will be treated separately.

6 Periodic Structure and Logarithmic Decomposition

The integrals introduced in the previous sections combine a logarithmic transformation with a fractional component. Their evaluation relies on the interaction between these two structures. This section develops the general decomposition that underlies all subsequent analytic methods.

6.1 Logarithmic Lattice Decomposition

Let

$$I_Q = \int_0^1 \{\ln(x)\}^\beta \, P_{\alpha,n}(x) Q(x) \, \mathrm{d}x$$

Using the logarithmic partition $(0,1) = \bigcup_{k=0}^{\infty} \left(e^{-(k+1)}, e^{-k}\right]$, we obtain

$$I_Q = \sum_{k=0}^{\infty} \int_{e^{-(k+1)}}^{e^{-k}} \{\ln(x)\}^\beta \, P_{\alpha,n}(x) Q(x) \, \mathrm{d}x$$

6.2 Reduction to the Unit Interval

On each interval, define

$$y = -\ln(x) - k,$$

where $0 \le y < 1$. Then,

$$x = e^{-(k+y)}, \qquad \mathrm{d}x = -e^{-(k+y)} \, \mathrm{d}y$$

Lemma 6.2.1 (Unit Interval Reduction). *Using substitutions provided, the integral becomes*

$$I_Q = \sum_{k=0}^{\infty} \int_0^1 \left\{\ln\left(e^{-(k+y)}\right)\right\}^\beta P_{\alpha,n}\left(e^{-(k+y)}\right) Q\left(e^{-(k+y)}\right) e^{-(k+y)} \, \mathrm{d}y$$

The **key structural outcome** is that all integrals of that particular class reduce to:

- A discrete sum over k;
- And an integral over $y \in [0,1)$.

6.3 Separation of Variables

We now isolate the dependence on k and y.

Lemma 6.3.1 (Kernel Separation). *For the canonical kernel,*

$$P_{\alpha,n}\left(e^{-(k+y)}\right) = \frac{(-1)^{n-1}}{\Gamma(n)} e^{-\alpha(k+y)}(k+y)^{n-1}$$

The integral is

$$I_Q = \frac{(-1)^{n-1}}{\Gamma(n)} \sum_{k=0}^{\infty} \int\limits_0^1 \left\{\ln\left(e^{-(k+y)}\right)\right\}^{\beta} e^{-\alpha(k+y)}(k+y)^{n-1} Q\left(e^{-(k+y)}\right) \mathrm{d}y$$

6.4 Periodic Structure (Abstract Form)

The expression $\left\{\ln\left(e^{-(k+y)}\right)\right\}$ depends on how the fractional part is defined for negative arguments. Rather than fixing a representation, we treat this term abstractly.

Definition 6.4.1 (Periodic Lift). *Let $F(t)$ denote a function, satisfying periodicity*

$$F(t+1) = F(t)$$

We interpret the fractional logarithmic structure as a periodic function evaluated at

$$t = \ln(x)$$

Property 6.4.1. *Under the change of variables,*

$$t = \ln\left(e^{-(k+y)}\right) = -(k+y)$$

so that

$$F(t) = F(-(k+y)).$$

This separates the problem into a **periodic function** in $y \pmod 1$ and a **non-periodic kernel** in $(k+y)$.

6.5 General Evaluation Principle

The previous results lead to a general principle.

Theorem 6.5.1 (Master Decomposition). *Let*

$$I_Q = \int\limits_0^1 \{\ln(x)\}^{\beta} P_{\alpha,n}(x) Q(x)\, \mathrm{d}x$$

Then,

$$I_Q = \sum_{k=0}^{\infty} \int_0^1 F(-(k+y))W(-(k+y))\,\mathrm{d}y$$

where $F(t)$ encodes the fractional part structure, and

$$W(t) = \frac{(-1)^{n-1}}{\Gamma(n)} t^{n-1} e^{-\alpha t} Q(e^{-t})$$

is the analytic kernel.

6.6 Baseline Integral

Define

$$I(\alpha, \beta, n) = \frac{(-1)^{n-1}}{\Gamma(n)} \int_0^1 \{\ln(x)\}^{\beta}\, x^{\alpha-1} \ln^{n-1}(x)\,\mathrm{d}x$$

where $Q(x) = 1$. This class, which serves as the primary object of study, is equivalent to

$$I(\alpha, \beta, n) = \int_{(0,1)^n} \left\{ \ln \left(\prod_{i=1}^{n} x_i \right) \right\}^{\beta} \left(\prod_{i=1}^{n} x_i \right)^{\alpha-1} \mathrm{d}x_1 \cdots \mathrm{d}x_n$$

More general integrals, involving $Q(x)$, will be treated as extensions of this case.

6.7 Feynman as the Primary Mechanism

From Section 4, recall that

$$I(\alpha, \beta, n) = \frac{(-1)^{n-1}}{\Gamma(n)} J(\alpha, \beta)$$

where

$$J(\alpha, \beta) = \int_0^1 \{\ln(x)\}^{\beta}\, x^{\alpha-1}\,\mathrm{d}x$$

6.8 Representation Dependence

The expression does not admit a unique analytic form without specifying a convention for the fractional part on negative arguments as noted in Section 2.

Remark. Different representations of the fractional part lead to a different intermediate forms of the integral $J(\alpha, \beta)$ and consequently $I(\alpha, \beta, n)$. Depending on the method chosen, these representations may produce:

- Series expansions;
- Finite algebraic expressions;
- Or hybrid integral-series forms;

6.9 Evaluation Framework

Given the decomposition developed earlier, we may write

$$J(\alpha, \beta) = \sum_{k=0}^{\infty} \int_0^1 \left\{\ln\left(e^{-(k+y)}\right)\right\}^{\beta} e^{-\alpha(k+y)}\, \mathrm{d}y$$

At this stage:

- No simplification of the fractional part is imposed;
- No expansion is assumed;
- And no special functions are introduced;

6.10 Consequences for Methods

Any evaluation must act on the expression $\left\{\ln\left(e^{-(k+y)}\right)\right\}$. The choice of representation determines:

- Whether the k-sum is tractable;
- Whether the y-integral simplifies;
- And whether the closed forms or series arise.

Observation. The algebraic expansion of terms, such as $(k+y)^{n-1}$, is independent of the fractional structure, but its usefulness depends on the compatibility of the chosen representation of the fractional part. Thus, algebraic methods may be effective for certain classes of integrals, but not uniformly across all kernels.

6.11 Outlook

The subsequent sections will develop specific representations of the fractional logarithmic component and apply them to the base integral $J(\alpha, \beta)$. Each representation leads to a distinct evaluation pathway, and no single method is universally optimal.

7 Evaluation of the Baseline Integral

We now evaluate the baseline integral introduced in Section 6. Throughout this section, we adopt the standard kernel normalization

$$J(\alpha, \beta) = \int_0^1 \{\ln(x)\}^\beta x^{\alpha-1} \, \mathrm{d}x$$

where $\Re(\alpha) > 0$. This integral serves as the generating object for the family

$$I(\alpha, \beta, n) = \frac{(-1)^{n-1}}{\Gamma(n)} \frac{\partial^{n-1}}{\partial \alpha^{n-1}} J(\alpha, \beta)$$

7.1 Logarithmic Reduction

Using the substitution $t = -\ln(x)$, we have $x = e^{-t}$ and $\mathrm{d}x = -e^{-t} \, \mathrm{d}t$. Altogether, this yields

$$J(\alpha, \beta) = \int_{\mathbb{R}^+} \{-t\}^\beta e^{-\alpha t} \, \mathrm{d}t$$

Lemma 7.1.1 (Laplace Form). *The baseline integral admits the representation*

$$J(\alpha, \beta) = \int_{\mathbb{R}^+} \Phi(t) e^{-\alpha t} \, \mathrm{d}t$$

where $\Phi(t) = \{-t\}^\beta$.

This expresses the problem as a **Laplace transform of a periodic-type function.**

7.2 Local Fractional Identities on the Unit Interval

Let $t = k + y$, where k is a non-negative integer, and $0 \leq y < 1$. Then, the evaluation of $\{-(k+y)\}$ reduces entirely to the behavior of the fractional part of the unit interval.

Proposition 7.2.1 (Local Fractional Representations). *For* $0 \leq y < 1$*, the fractional part of* $-(k+y)$ *admits the following representation:*

- ***Knuth Convention:****[4]* $\{-(k+y)\} = 1 - y$
- ***Fourier / Symmetric Convention:*** $\{-(k+y)\} = -y$

Proof. Since k is an integer, we have

$$-(k+y) = -(k+1) + (1-y),$$

which yields the positive-range representation

$$\{-(k+y)\} = 1 - y.$$

Alternatively, under the symmetric convention, where the fractional part preserves sign for negative arguments, we obtain

$$\{-(k+y)\} = -y$$

$\square$

7.3 Consequence: Exact Separation

Substituting into the baseline integral,

$$J(\alpha,\beta) = \sum_{k=0}^{\infty} \int_0^1 \{-(k+y)\}^{\beta} \, e^{-\alpha(k+y)} \, \mathrm{d}y$$

We obtain two fully separable forms.

Theorem 7.3.1 (Separation Formula Under Local Representations)**.** *Under either convention, the integral separates as*

$$J(\alpha,\beta) = \left(\sum_{k=0}^{\infty} e^{-\alpha k}\right) \int_0^1 H(\beta; y) e^{-\alpha y} \, \mathrm{d}y$$

where

$$H(\beta; y) = \begin{cases} (-y)^{\beta}, & \textit{Fourier} \\ (1-y)^{\beta}, & \textit{Knuth} \end{cases}$$

The above term arrangement consequently yields the following corollary.

Corollary 7.3.1. *For $\alpha > 0$, by power series,*

$$\sum_{k=0}^{\infty} e^{-\alpha k} = \frac{1}{1 - e^{-\alpha}}$$

and therefore

$$J(\alpha,\beta) = \frac{1}{1-e^{-\alpha}} \int_0^1 H(\beta; y) e^{-\alpha y} \, \mathrm{d}y$$

Structural Interpretation. This result shows that:

- The difficulty in Section 6 was not fundamental.
- It arose from keeping the fractional part in an unresolved form.
- Once reduced to $0 \le y < 1$, the structure becomes completely explicit.

Key Insight. The entire problem reduces to evaluating

$$\int_0^1 H(\beta; y) e^{-\alpha y} \, dy$$

with

$$H(\beta; y) = \begin{cases} (-y)^\beta, & \text{Fourier} \\ (1-y)^\beta, & \text{Knuth} \end{cases}$$

7.4 Feynman Completion

Once we evaluate

$$K(\alpha, \beta) = \int_0^1 H(\beta; y) e^{-\alpha y} \, dy$$

in terms of α and β, we substitute that into

$$I(\alpha, \beta, n) = \frac{(-1)^{n-1}}{\Gamma(n)} \frac{\partial^{n-1}}{\partial \alpha^{n-1}} \left[\frac{1}{1 - e^{-\alpha}} \int_0^1 H(\beta; y) e^{-\alpha y} \, dy \right].$$

7.5 Explicit Evaluation of the Unit-Cell Integral

We evaluate

$$J(\alpha, \beta) = \frac{1}{1 - e^{-\alpha}} \int_0^1 H(\beta; y) e^{-\alpha y} \, dy$$

By inspection, since $H(\beta; y)$ is composed of similar forms, we start with the Fourier convention before the Knuth convention.

7.5.1 Fourier / Symmetric Convention

We compute

$$K_F(\alpha, \beta) = \int_0^1 (-y)^\beta e^{-\alpha y} \, dy = (-1)^\beta \int_0^1 y^\beta e^{-\alpha y} \, dy \tag{1}$$

Lemma 7.5.1. *For integers $\beta \geq 0$,*

$$\int_0^1 y^\beta e^{-\alpha y}\,\mathrm{d}y = \frac{\beta!}{\alpha^{\beta+1}}\left(1 - e^{-\alpha}\sum_{k=0}^{\beta}\frac{\alpha^k}{k!}\right)$$

Proof. By integration by parts, we have

$$R(\beta) = -\frac{e^{-\alpha}}{\alpha} + \frac{\beta}{\alpha}R(\beta-1).$$

With

$$R(0) = \int_0^1 e^{-\alpha y}\,\mathrm{d}y = \frac{1-e^{-\alpha}}{\alpha}$$

Iterations reveal the general expansion:

$$R(\beta) = \frac{\beta!}{\alpha^{\beta+1}} - e^{-\alpha}\sum_{k=0}^{\beta}\frac{\beta!}{k!\alpha^{\beta-k+1}}$$

Alternatively, by re-indexing the summation with $j = \beta - k$, the expression can be written as

$$R(\beta) = \int_0^1 y^\beta e^{-\alpha y}\,\mathrm{d}y = \frac{\beta!}{\alpha^{\beta+1}}\left(1 - e^{-\alpha}\sum_{k=0}^{\beta}\frac{\alpha^k}{k!}\right)$$

□

Therefore,

$$J_F(\alpha,\beta) = \frac{(-1)^\beta \beta!}{\alpha^{\beta+1}(1-e^{-\alpha})}\left(1 - e^{-\alpha}\sum_{k=0}^{\beta}\frac{\alpha^k}{k!}\right) \tag{2}$$

7.5.2 Knuth Convention

Next, we compute

$$K_K(\alpha,\beta) = \int_0^1 (1-y)^\beta e^{-\alpha y}\,\mathrm{d}y.$$

With the substitution $u = 1 - y$, then

$$K_K(\alpha,\beta) = e^{-\alpha}\int_0^1 u^\beta e^{\alpha u}\,\mathrm{d}u$$

Deploying Equation 1 reveals that

$$J_K(\alpha,\beta) = \frac{\beta! e^{-\alpha}}{\alpha^{\beta+1}(1-e^{-\alpha})}\left(-1 + e^{\alpha}\sum_{k=0}^{\beta}\frac{(-\alpha)^k}{k!}\right) \tag{3}$$

7.6 Leibniz Evaluation (Direct Form)

To evaluate

$$I(\alpha,\beta,n) = \frac{(-1)^{n-1}}{\Gamma(n)}\frac{\partial^{n-1}}{\partial\alpha^{n-1}}J(\alpha,\beta),$$

where J is either J_F or J_K, we deploy the **general Leibniz rule**.

Theorem 7.6.1 (General Leibniz Rule)**.** *Let $f_1,\dots,f_m$ denote m differentiable functions. Then*

$$(f_1\cdots f_m)^{(n)} = \sum_{k_1+k_2+\cdots+k_m=n}\binom{n}{k_1,k_2,\dots,k_m}\prod_{t=1}^{m}(f_t)^{(k_t)}$$

where

$$\binom{n}{k_1,k_2,\dots,k_m} = \frac{n!}{k_1!k_2!k_m!}$$

are the multinomial coefficients.

7.6.1 Fourier Case

Let

$$\begin{aligned} f_1(\alpha) &= \left(1-e^{-\alpha}\right)^{-1} = \frac{1}{2}\left(\coth\left(\frac{\alpha}{2}\right)+1\right) \\ f_2(\alpha) &= \alpha^{-(\beta+1)} \\ f_3(\alpha) &= 1 - e^{-\alpha}\sum_{k=0}^{\beta}\frac{\alpha^k}{k!} \end{aligned}$$

where then

$$I_F(\alpha,\beta,n) = \frac{(-1)^{\beta+n-1}\beta!}{\Gamma(n)}\sum_{n_1+n_2+n_3=n-1}\binom{n-1}{n_1,n_2,n_3}\left(\prod_{i=1}^{3}(f_i(\alpha))^{n_i}\right) \tag{4}$$

By telescoping, the derivatives are explicitly

$$
\begin{aligned}
(f_1(\alpha))^{n_1} =& \frac{1}{2}\left(\left(\frac{\partial^{n_1}}{\partial\alpha^{n_1}}1\right) - \frac{2e^{\alpha}}{(1-e^{2\alpha})^{n_1+1}}\sum_{m=0}^{n_1-1}\left\langle {n_1 \atop m} \right\rangle e^{2m\alpha}\right) \\
=& (-1)^{n_1}\sum_{m=1}^{n_1} m! S(n_1, m)\frac{e^{-m\alpha}}{(1-e^{-\alpha})^{m+1}} \\
(f_2(\alpha))^{(n_2)} =& \alpha^{-(\beta+1+n_2)}(-\beta-n_2)_{n_2} \\
(f_3(\alpha))^{(n_3)} =& e^{-\alpha}\sum_{m=0}^{n_3-1}\frac{(-1)^m}{(\beta-n_3+1+m)!}\binom{n_3-1}{m}\alpha^{\beta-n_3+1+m}
\end{aligned}
$$

where $S(n,k)$ denotes the Stirling number of the second kind, and $(n)_k$ denotes the Pochhammer symbol. Explicitly, the Stirling number of the second kind[3] is

$$
S(n,k) = \sum_{i=0}^{k}\frac{(-1)^{k-i}i^n}{(k-i)!i!}
$$

7.6.2 Knuth Case

Likewise, let

$$
\begin{aligned}
g_1(\alpha) =& (1-e^{\alpha})^{-1} \\
g_2(\alpha) =& \alpha^{-(\beta+1)} \\
g_3(\alpha) =& -e^{\alpha} + \sum_{k=0}^{\beta}\frac{(-\alpha)^k}{k!}
\end{aligned}
$$

Then,

$$
I_K(\alpha,\beta,n) = \frac{(-1)^{n-1}\beta!}{\Gamma(n)}\sum_{n_1+n_2+n_3=n-1}\binom{n-1}{n_1,n_2,n_3}\left(\prod_{i=1}^{3}(g_i(\alpha))^{n_i}\right) \tag{5}
$$

From the previous subsubsection,

$$
(g_1(\alpha))^{(n_1)} = (-1)^{n_1}\sum_{m=1}^{n_1} m! S(n_1,m)\frac{e^{-m\alpha}}{(1-e^{-\alpha})^{m+1}}
$$

In addition,

$$
(g_2(\alpha))^{(n_2)} = (f_2(\alpha))^{(n_2)} = \alpha^{-(\beta+1+n_2)}(-\beta-n_2)_{n_2}
$$

For the remaining n_3-th derivative,

$$(g_3(\alpha))^{(n_3)} = (-1)^{n_3}\left[-(-1)^{n_3}e^{\alpha} + \sum_{k=0}^{\beta-n_3}\frac{(-\alpha)^k}{k!}\right]$$

After reduction to the unit interval, all complexity is confined to a finite product of three elementary components; the full evaluation follows directly from the multinomial Leibniz rule.

8 Role of the Feynman Method and Structural Choice

This section touches on the approach taken between Section 3 and Section 7. Please note that the kind of approaches considered depends largely on the integral.

8.1 Purpose of the Method

The use of the Feynman method is not intended to:

- Replace other analytic techniques,
- Or eliminate the appearance of special functions in general.

It is rather to provide a **natural and efficient evaluation route**[2] for the baseline class of integrals.

8.2 Origin of the Integral Structure

The baseline integral arises from the substitution $x = e^{-y}$, which transforms expressions, involving $\ln(x)$ into:

- Exponential factors $e^{-\alpha y}$;
- Polynomial factors in y;
- And bounded fractional components.

Key Observation. The resulting structure

$$e^{-\alpha y} \cdot y^m$$

is not artificially imposed, but **inherent to the logarithmic transformation**.

8.3 Why Feynman Fits Naturally

The factor $\ln^{n-1}(x)$ is generated via

$$\frac{\partial^{n-1}}{\partial \alpha^{n-1}} x^{\alpha-1},$$

which aligns directly with the exponential structure obtained after substitution. Following are some consequences of using the Feynman method:

- No inversion of sums or transforms is introduced.
- The resulting expression remain with a closed, finitary-sum algebraic-exponential form.

Remark. The auxiliary representations or special functions can occur in the evaluation as seen in Section 7.

8.4 On the Use of Gamma-Type Representations

It is possible to approach the same class of integrals, using:

- Gamma distribution kernels
- Mellin-type transforms

However, such approaches typically lead to:

- Additional layers of transformation
- More complex intermediate expressions

Important Clarification: The choice to avoid Gamma-based methods here is motivated by the structural simplicity, not by their inherent difficulty.

8.5 Appearance of Special Functions

Special functions may arise, depending on how the integral is manipulated. Two key points are that:

1. They are not artificially avoided.
2. But they are also not guaranteed to appear.

In some formulations, the integral may lead to structures, including:

- Hypergeometric-type
- Multi-zeta

In the present formulation, the same quantities reduce to:

- Finite sums
- Elementary exponential expressions

Key Principle. The appearance of special functions reflects the chosen representation, not necessarily the intrinsic complexity of the integral.

8.6 Multiple Valid Evaluation Routes

There is no single "correct" method. Depending on the context, one may use methods, like:

- Feynman differentiation
- Direct expansion
- Integral transforms
- Probabilistic interpretations

For the baseline class, the Feynman route provides a particularly direct and controlled evaluation.

8.7 Scope and Limitations

The effectiveness of this method depends on:

- The compatibility of the kernel with parameter differentiation
- The bounded nature of the fractional component

Consequently, some extensions remain elementary, whereas others may naturally introduce special functions. From Section 3 to Section 7, the results show that a natural class of fractional-part integrals admits explicit evaluation through a structurally aligned method.

Balanced Statement. The method does not eliminate alternative approaches, but provides a framework in which the integral simplifies without requiring additional analytic machinery.

9 Multi-Variable Extension and Product Structure

This section formalizes important concepts introduced in previous sections.

9.1 Multi-Variable Extension and Product Structure

Sections 3 - 7 developed a complete evaluation for the one-dimensional integral $I(\alpha, \beta, n)$. However, the logarithmic structure suggests a natural extension:

$$\ln(x_1 x_2 \cdots x_n) = \sum_{i=1}^{n} \ln(x_i)$$

The key idea here is that instead of treating higher powers via differentiation alone, we can generate structures through independent variables.

9.2 Product of Uniform Variables

Consider

$$I(\alpha, \beta, n) = \int\limits_{(0,1)^n} \left\{ \ln\left(\prod_{i=1}^{n} x_i \right) \right\}^{\beta} \left(\prod_{i=1}^{n} x_i \right)^{\alpha-1} \mathrm{d}x_1 \ldots \mathrm{d}x_n$$

Given the integral contains

$$\ln\left(\prod_{i=1}^{n} x_i \right)$$

There are several ways to treat an integral. From Section 3, by letting $X_1, X_2, \ldots, X_n$ be random independent variables uniform on the unit interval $(0, 1)$, we set

$$Z = X_1 X_2 \cdots X_n$$

which transforms to the one-dimensional case. Alternatively, we can set

$$-\ln(Z) = \sum_{i=1}^{n} (-\ln X_i)$$

Since each $-\ln X_i \sim \mathrm{Exp}(1)$, where $\mathrm{Exp}(\lambda)$ denotes the exponential distribution, then

$$-\ln(Z) \sim \Gamma(n, 1)$$

where $\Gamma(\alpha, \beta)$ denotes the Gamma distribution. While this connects to the Gamma distribution, it only serves as a structural reference.

9.3 Reduction to One-Dimensional Form

By setting $Z = x_1 \cdots x_n$, we obtain the general form:

$$\int\limits_{(0,1)^n} f(x_1 \cdots x_n)\,\mathrm{d}x_1 \ldots \mathrm{d}x_n = \int\limits_0^1 f(x) K_n(x)\,\mathrm{d}x$$

where

$$K_n(x) = \frac{(-1)^{n-1}}{\Gamma(n)} \ln^{n-1}(x)$$

Thus, for the baseline integral, we have

$$I(\alpha, \beta, n) = \frac{(-1)^{n-1}}{\Gamma(n)} \int\limits_0^1 \{\ln(x)\}^{\beta}\, x^{\alpha-1} \ln^{n-1}(x)\,\mathrm{d}x$$

By probability theory, we have shown the equivalence of the **multi-variable formulation** and the **one-dimensional formulation** of the same object.

9.4 Extension to General Classes

We can extend $I(\alpha, \beta, n)$ to

$$I_Q = \int\limits_{(0,1)^n} \{\ln(x_1 \cdots x_n)\}^{\beta}\, Q(x_1 \cdots x_n) \left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} \mathrm{d}x_1 \ldots \mathrm{d}x_n$$

which can be reduced to

$$\int\limits_0^1 \{\ln(x)\}^{\beta}\, Q(x) x^{\alpha-1} K_n(x)\,\mathrm{d}x$$

Consequently, all $Q(x)$ from Section 5 carry over directly. The logarithmic sum $\sum\limits_{i=1}^{n} \ln(x_i)$ acts as an additive convolution under the transformation $y = -\ln(z)$, yielding the kernel $K_n(x)$. This kernel arises naturally as:

- Repeated addition of exponential variables
- Not an imposed analytic device

For this kind of $Q(x)$ classes, like $Q(x) = 1$, the integral admits three different representations:

- One-dimensional form
- Multi-variable form
- Feynman form by differentiation with respect to the parameter α

which will be covered in next sections.

Final Perspective. The apparent increase in dimensionality reflects a change in representation, not in intrinsic difficulty.

10 Admissibility of $Q(x)$

10.1 Motivation and Scope

Up to this point, we have focused on integrals of the form

$$I(\alpha, \beta, n) = \frac{(-1)^{n-1}}{\Gamma(n)} \int_0^1 \{\ln(x)\}^\beta \, x^{\alpha-1} \ln^{n-1}(x) \, \mathrm{d}x$$

where $Q(x) = 1$. The structure of this integral allows for systematic evaluation via substitution and differentiation with respect to the parameter α. We now extend this framework to the more general class

$$I(\alpha, \beta, n; Q) = \frac{(-1)^{n-1}}{\Gamma(n)} \int_0^1 \{\ln(x)\}^\beta \, x^{\alpha-1} \ln^{n-1}(x) \, \mathrm{d}x$$

where $Q(x)$ is an additional factor. As discussed in Section 5, the integral may be interpreted as a product of following components:

$$I(\alpha, \beta, n; Q) = \frac{(-1)^{n-1}}{\Gamma(n)} \int_0^1 \{\ln(x)\}^\beta \, P_{\alpha,n}(x) Q(x) \, \mathrm{d}x,$$

where $P_{\alpha,n}(x) = x^{\alpha-1} \ln^{n-1}(x)$. Rather than imposing rigid structural constraints on $Q(x)$, we adopt the following guiding principle:

> *The factor $Q(x)$ is admissible if the resulting integral remains well-defined and compatible with the evaluation methods developed for the baseline case.*

In particular, admissibility is determined by whether the inclusion of $Q(x)$ preserves:

- The convergence of the interval on $(0, 1)$,
- The validity of parameter differentiation (Feynman method),
- And tractability after the logarithmic substitution $x = e^{-y}$.

With this perspective, the structure introduced in Section 7 extends naturally. Specifically, we may write

$$I(\alpha, \beta, n; Q) = \frac{(-1)^{n-1}}{\Gamma(n)} \left[\frac{\partial^{n-1}}{\partial \alpha^{n-1}} J(\alpha, \beta; Q) \right]$$

where

$$J(\alpha,\beta;Q)=\int_0^1 \{\ln(x)\}^{\beta}\,x^{\alpha-1}Q(x)\,\mathrm{d}x.$$

This formulation isolates the role of the parameter α and reduces the problem to the evaluation of $J(\alpha,\beta;Q)$, after which higher-order logarithmic factors are recovered through differentiation.

The purpose of this section is to demonstrate how different choices of $Q(x)$ interact with the fractional part structure and to identify situations in which the integral remains amenable to explicit evaluation.

10.2 The Separability Condition

In Section 6 and 7, the explicit evaluation of the baseline integral relied entirely on the property that the transformed interval $t=k+y$ decoupled the fractional part from the integer summation. For a generalized factor $Q(x)$, we apply the logarithmic substitution $x=e^{-(k+y)}$, which yields:

$$J(\alpha,\beta;Q)=\sum_{k=0}^{\infty}e^{-\alpha k}\int_0^1 H(\beta;y)e^{-\alpha y}Q\left(e^{-(k+y)}\right)\,\mathrm{d}y$$

where $H(\beta;y)$ is the representation of the fractional part of the unit interval. The admissibility of $Q(x)$ - and the existence of a closed-form evaluation - depends strictly on the functional behavior of $Q\left(e^{-(k+y)}\right)$. The next section categorizes admissible extensions into manageable classes.

11 Extensions of $Q(x)$

While the previous section established the foundational admissibility of the $Q(x)$ kernel, the true analytical depth of these logarithmic fractional integrals emerges when we relax the constraints of the unit cube and explore varied periodic definitions. Section 11 extends the $Q(x)$ framework into higher-dimensional product structures and examines the implications of adopting alternative fractional part definitions. A primary focus is placed on the Knuth case, where the treatment of negative arguments departs from standard periodic extensions. By analyzing these simpler, yet distinct, classes, we can isolate the mechanical differences in how the fractional part interacts with logarithmic kernels, ultimately revealing the sensitivity of the associated constants to the underlying definition of the fractional operator.

11.1 Classes, Involving x

Suppose that $Q(x) = x^\gamma$, where $\Re(\alpha + \gamma) > 0$. The steps to evaluate the integral below mimic steps from Section 7.

$$J(\alpha, \beta; x^\gamma) = \frac{1}{1 - e^{-(\alpha+\gamma)}} \int_0^1 \{-y\}^\beta \, e^{-y(\alpha+\gamma)} \, \mathrm{d}y$$

11.1.1 Fourier Case

Using the form from Equation 2

$$J_F(\alpha, \beta; x^\gamma) = \frac{(-1)^\beta \beta!}{(\alpha+\gamma)^{\beta+1} \left(1 - e^{-(\alpha+\gamma)}\right)} \left(1 - e^{-(\alpha+\gamma)} \sum_{k=0}^{\beta} \frac{(\alpha+\gamma)^k}{k!}\right)$$

Therefore, from Equation 4,

$$I_F(\alpha, \beta, n; x^\gamma) = \frac{(-1)^{\beta+n-1}\beta!}{\Gamma(n)} \sum_{n_1+n_2+n_3=n-1} \binom{n-1}{n_1, n_2, n_3} \left(\prod_{i=1}^{3} (f_i(\alpha+\gamma))^{(n_i)}\right)$$

where

$$\begin{aligned}
(f_1(\alpha+\gamma))^{(n_1)} =& (-1)^{n_1} \sum_{m=1}^{n_1} m! S(n_1, m) \frac{e^{-m(\alpha+\gamma)}}{(1 - e^{-(\alpha+\gamma)})^{m+1}} \\
(f_2(\alpha+\gamma))^{(n_2)} =& (\alpha+\gamma)^{-(\beta+1+n_2)} \, (-\beta - n_2)_{n_2} \\
(f_3(\alpha+\gamma))^{(n_3)} =& e^{-(\alpha+\gamma)} \sum_{m=0}^{n_3-1} \frac{(-1)^m}{(\beta - n_3 + 1 + m)!} \binom{n_3 - 1}{m} (\alpha+\gamma)^{\beta - n_3 + 1 + m}
\end{aligned}$$

11.1.2 Knuth Case

In the similar manner, from Equation 3,

$$J_K(\alpha, \beta; x^\gamma) = \frac{\beta! e^{-(\alpha+\gamma)}}{(\alpha+\gamma)^{\beta+1}\left(1 - e^{-(\alpha+\gamma)}\right)}\left(-1 + e^{\alpha+\gamma}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(\alpha+\gamma)^k\right)$$

Therefore, from Equation 5

$$I_K(\alpha, \beta, n; x^\gamma) = \frac{(-1)^{n-1}\beta!}{\Gamma(n)}\sum_{n_1+n_2+n_3=n-1}\binom{n-1}{n_1, n_2, n_3}\left(\prod_{i=1}^{3}(g_i(\alpha+\gamma))^{(n_i)}\right)$$

where

$$\begin{aligned}
(g_1(\alpha+\gamma))^{(n_1)} =& (-1)^{n_1}\sum_{m=1}^{n_1} m! S(n_1, m)\frac{e^{-m(\alpha+\gamma)}}{(1 - e^{-(\alpha+\gamma)})^{m+1}} \\
(g_2(\alpha+\gamma))^{(n_2)} =& (\alpha+\gamma)^{-(\beta+1+n_2)}(-\beta - n_2)_{n_2} \\
(g_3(\alpha+\gamma))^{(n_3)} =& (-1)^{n_3}\left[-(-1)^{n_3}e^{\alpha+\gamma} + \sum_{k=0}^{\beta-n_3}\frac{(\alpha+\gamma)^k(-1)^k}{k!}\right]
\end{aligned}$$

11.2 Classes, Involving $\{\ln(x)\}$

Suppose that $Q(x) = \{\ln(x)\}^\gamma$, where

- $(\beta+\gamma) \geq 0$
- β and γ are both non-negative integers

Then,

$$J(\alpha, \beta; \{\ln(x)\}^\gamma) = \frac{1}{1 - e^{-\alpha}}\int_0^1 \{-y\}^{\beta+\gamma} e^{-\alpha y}\, dy$$

The following subsubsections contain complete closed forms of their respective derivatives. Since $\{\ln(x)\}^\gamma$ is involved, we use Equation 2 - 5 from Section 7 to evaluate the integral.

11.2.1 Fourier Case

In the similar manner,

$$J_F(\alpha, \beta; \{\ln(x)\}^\gamma) = \frac{(-1)^{\beta+\gamma}(\beta+\gamma)!}{(\alpha)^{\beta+\gamma+1}(1 - e^{-\alpha})}\left(1 - e^{-\alpha}\sum_{k=0}^{\beta+\gamma}\frac{(\alpha)^k}{k!}\right)$$

Therefore,

$$I_F(\alpha,\beta,n;\{\ln(x)\}^\gamma) = \frac{(-1)^{\beta+\gamma+n-1}\,(\beta+\gamma)!}{\Gamma(n)} \sum_{n_1+n_2+n_3=n-1} \binom{n-1}{n_1,n_2,n_3} \left(\prod_{i=1}^{3}(f_i(\alpha))^{(n_i)}\right)$$

where

$$\begin{aligned}
(f_1(\alpha))^{(n_1)} =&(-1)^{n_1}\sum_{m=1}^{n_1} m!S(n_1,m)\frac{e^{-m\alpha}}{(1-e^{-\alpha})^{m+1}}\\
(f_2(\alpha))^{(n_2)} =&\alpha^{-(\beta+\gamma+1+n_2)}(-\beta-\gamma-n_2)_{n_2}\\
(f_3(\alpha))^{(n_3)} =&e^{-\alpha}\sum_{m=0}^{n_3-1}\frac{(-1)^m}{(\beta+\gamma-n_3+1+m)!}\binom{n_3-1}{m}\alpha^{\beta+\gamma-n_3+1+m}
\end{aligned}$$

11.2.2 Knuth Case

In the similar manner,

$$J_F(\alpha,\beta,\{\ln(x)\}^\gamma) = \frac{(\beta+\gamma)!e^{-\alpha}}{\alpha^{\beta+\gamma+1}(1-e^{-\alpha})}\left(-1+e^{\alpha}\sum_{k=0}^{\beta+\gamma}\frac{(-\alpha)^k}{k!}\right)$$

Therefore,

$$I_K(\alpha,\beta,n;\{\ln(x)\}^\gamma) = \frac{(-1)^{n-1}(\beta+\gamma)!}{\Gamma(n)} \sum_{n_1+n_2+n_3=n-1} \binom{n-1}{n_1,n_2,n_3} \left(\prod_{i=1}^{3}(g_i(\alpha))^{(n_i)}\right)$$

where

$$\begin{aligned}
(g_1(\alpha))^{(n_1)} =&(-1)^{n_1}\sum_{m=1}^{n_1} m!S(n_1,m)\frac{e^{-m\alpha}}{(1-e^{-\alpha})^{m+1}}\\
(g_2(\alpha))^{(n_2)} =&(\alpha)^{-(\beta+1+n_2)}(-\beta-n_2)_{n_2}\\
(g_3(\alpha))^{(n_3)} =&(-1)^{n_3}\left[-(-1)^{n_3}e^{\alpha}+\sum_{k=0}^{\beta-n_3}\frac{(\alpha)^k(-1)^k}{k!}\right]
\end{aligned}$$

11.3 Classes, Involving $\{-\ln(x)\}$

Suppose that $Q(x) = \{-\ln(x)\}^\gamma$, where

- $(\beta+\gamma) \geq 0$

- β and γ are both non-negative integers

In the previous case, for $0 < x < 1$, $\ln(x) < 0$, where the definition of $\{\ln(x)\}$ depends on the chosen convention for negative arguments. On the other hand, for $0 < x < 1$, $-\ln(x) > 0$, where $\{-\ln(x)\}$ follows the definition for positive arguments. Therefore,

$$J(\alpha, \beta; \{-\ln(x)\}^\gamma) = \frac{1}{1-e^{-\alpha}} \int_0^1 \{-y\}^\beta \{y\}^\gamma e^{-\alpha y} \, \mathrm{d}y$$

11.3.1 Fourier Case

By definition, for negative real values y, $\{-y\} = -\{y\}$, which transforms the integral into

$$J(\alpha, \beta; \{-\ln(x)\}^\gamma) = \frac{(-1)^\beta}{1-e^{-\alpha}} \int_0^1 y^{\gamma+\beta} e^{-\alpha y} \, \mathrm{d}y$$

Hence,

$$J_F(\alpha, \beta; \{-\ln(x)\}^\gamma) = \frac{(-1)^\beta (\beta+\gamma)!}{\alpha^{\beta+\gamma}(1-e^{-\alpha})} \left(1 - e^{-\alpha} \sum_{k=0}^{\beta+\gamma} \frac{\alpha^k}{k!}\right)$$

Therefore,

$$I_F(\alpha, \beta, n; \{-\ln(x)\}^\gamma) \frac{(-1)^{\beta+n-1}(\beta+\gamma)!}{\Gamma(n)} \sum_{n_1+n_2+n_3=n-1} \binom{n-1}{n_1, n_2, n_3} \left(\prod_{i=1}^{3} (f_i(\alpha))^{(n_i)}\right)$$

where

$$\begin{aligned}
(f_1(\alpha))^{(n_1)} &= (-1)^{n_1} \sum_{m=1}^{n_1} m! S(n_1, m) \frac{e^{-m\alpha}}{(1-e^{-\alpha})^{m+1}} \\
(f_2(\alpha))^{(n_2)} &= \alpha^{-(\beta+\gamma+1+n_2)} (-\beta-\gamma-n_2)_{n_2} \\
(f_3(\alpha))^{(n_3)} &= e^{-\alpha} \sum_{m=0}^{n_3-1} \frac{(-1)^m}{(\beta+\gamma-n_3+1+m)!} \binom{n_3-1}{m} \alpha^{\beta+\gamma-n_3+1+m}
\end{aligned}$$

11.3.2 Knuth Case

By definition, for negative real values y, $\{-y\} = 1 - y$, which writes

$$J_K(\alpha,\beta;\{-\ln(x)\}^\gamma) = \frac{1}{1-e^{-\alpha}}\int\limits_0^1 (1-y)^\beta y^\gamma e^{-\alpha y}\,\mathrm{d}y$$

Using the binomial theorem, we have

$$\begin{aligned} J_K(\alpha,\beta;\{-\ln(x)\}^\gamma) =& \frac{1}{1-e^{-\alpha}}\sum_{l=0}^{\beta}\binom{\beta}{l}(-1)^l\int\limits_0^1 y^{l+\gamma}e^{-\alpha y}\,\mathrm{d}y \\ =& \frac{1}{1-e^{-\alpha}}\sum_{l=0}^{\beta}\left[\binom{\beta}{l}(-1)^l\frac{(l+\gamma)!}{\alpha^{l+\gamma+1}}\left(1-e^{-\alpha}\sum_{k=0}^{l+\gamma}\frac{\alpha^k}{k!}\right)\right] \end{aligned}$$

Therefore, following Section 11.2.1, the integral $I_K(\alpha,\beta;\{-\ln(x)\}^\gamma)$ is

$$\frac{1}{\Gamma(n)}\sum_{l=0}^{\beta}\left[\binom{\beta}{l}(-1)^{n+l-1}(l+\gamma)!\sum_{n_1+n_2+n_3=n-1}\binom{n-1}{n_1,n_2,n_3}\left(\prod_{i=1}^{3}(g_i(\alpha))^{(n_i)}\right)\right]$$

where

$$\begin{aligned} (g_1(\alpha))^{(n_1)} =& (-1)^{n_1}\sum_{m=1}^{n_1} m!S(n_1,m)\frac{e^{-m\alpha}}{(1-e^{-\alpha})^{m+1}} \\ (g_2(\alpha))^{(n_2)} =& \alpha^{-(l+\gamma+1+n_2)}\left(-l-\gamma-n_2\right)_{n_2} \\ (g_3(\alpha))^{(n_3)} =& e^{-\alpha}\sum_{m=0}^{n_3-1}\frac{(-1)^m}{(l+\gamma-n_3+1+m)!}\binom{n_3-1}{m}\alpha^{l+\gamma-n_3+1+m} \end{aligned}$$

Key Insight. The simplification of $I(\alpha,\beta,n;\{-\ln(x)\}^\beta)$ significantly depends on the chosen convention of the fractional part function. Fourier definition of a negative fractional part function simplifies to the integral, involving the monomial term and the exponential function. On the other hand, Knuth definition of a negative fractional part yields the integral, containing the Beta-type kernel $(1-y)^\beta y^\gamma$.

11.4 Conclusion

The extensions explored in this section underscore that the logarithmic fractional part function is far more than a local curiosity of the unit interval. By contrasting the standard evaluation with the Knuth definition, we have shown that the choice of fractional part for negative arguments—often overlooked in elementary contexts—fundamentally reconfigures the resulting series expansions and the convergence behavior of their coefficients. These variations suggest a broader taxonomy of logarithmic fractional integrals, where the structural 'drift' observed in the Knuth case provides a template for analyzing more complex, non-periodic kernels. As we move toward more generalized multi-variable extensions, these differences will serve as a benchmark for characterizing the limits of analytic continuity in logarithmic fractional systems.

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